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Yours truly
James Smith

THE

QUADRATURE AND GEOMETRY

OF THE

CIRCLE DEMONSTRATED

BY

JAMES SMITH

Liverpool
EDWARD HOWELL, CHURCH STREET
London
SIMPKIN, MARSHALL & CO., STATIONERS' HALL COURT
1872.
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ADDRESS TO THE READER.

THE following pages were in the hands of the printer when death terminated the earthly career of the Author. The notice below is extracted from the last edition (1868) of *Men of the Time*:—

"SMITH, JAMES, son of the late Joshua Smith, born in Liverpool, March 26th, 1805; at an early age entered a merchant's counting-house, where he remained seventeen years, when he commenced business on his own account, and was enabled to retire with a competency in 1855. He had during a long period studied Geometry and Mathematics, and devoted much attention to Mechanical experiments; the latter, for the purpose of facilitating mining operations; for the more perfect consumption of coal for steam and other purposes, but chiefly with a view of demonstrating whether the natural forces of air and water can or cannot be made to produce a primary motive power independent of chemical change. All the results of his various mechanical investigations have not as yet been made public, but his geometrical and mathematical researches have from time to time been published, and he lays claim to a discovery of much importance, no other than that of solving the problem of the true ratio of diameter to circumference in a circle, or, as it is familiarly termed, 'Squaring the Circle.' Mr. Smith's theory on this abstruse question has, however, been severely handled by mathematicians; among others by the late Dr. Whewell, and Lieut.-Gen. T. Perronet Thompson, but more especially by Professor de Morgan, who has taken much pains in the columns of the *Athenæum* to fix upon him the character of a 'paradoxer of unerasable reputation.' From the attacks of opponents to his theory, Mr. Smith has defended himself in his numerous published works on the Quadrature of the Circle.

"He was nominated by the Board of Trade to a seat at the Liverpool Local Marine Board, and has for some years been its chairman ; is a member of the Mersey Docks and Harbour Board, and of the Literary and Philosophical, the Polytechnic, and other literary and scientific associations of his native town."

The above short biographical notice presents to us a man of somewhat extraordinary character. The writer of this address had the privilege of being associated with Mr. Smith in the closest bonds of friendship for upwards of thirty years, therefore affording him ample opportunity for the formation of a correct estimate of the merits and qualities of the man. As a Liverpool merchant, the deceased gentleman was distinguished for shrewdness, honesty, and integrity. As a public man, he was pre-eminently distinguished for his clear judgment, retentive memory, soundness of his opinions, and his fearless, out-spoken, and independent dealing with all public questions which came under his notice. With such characteristics it would have been difficult, if not impossible, in the heat of debate, to avoid occasional collisions with his colleagues in office ; but, from long observation, the writer can fairly attribute such manifestations on his part to an excess of earnestness and zeal, instead of any desire to avoid the arguments, or wound the feelings of an adversary.

For nearly twenty years, Mr. Smith devoted a great part of his time to the gratuitous discharge of his public duties ; and his valuable services, especially as a member of the Mersey Docks and Harbour Board, will be long remembered in his native town. About two years ago, failing health compelled him to retire from public life.

In referring to Mr. Smith's mathematical and geometrical labours, the reader's attention is directed to the already published works of the author, in all of which he

challenges the recognised Mathematicians of the present day to refute his theory. It is with firm confidence, that the impartial reader, and faithful searcher after truth, will readily concede, that the various processes of reasoning, and the methods adopted by the author—should they not culminate in the reader's mind in the result contended for by Mr. Smith—yet, irrefragably demonstrate the erroneousness of the generally received opinion, viz. :— *It is proved beyond doubt, that the true and exact ratio of diameter to the circumference of a circle cannot be arithmetically given in finite numbers;* and so fix upon the Author's opponents the *onus probandi* of their own assumption—which, unless supported by other and stronger materials than those hitherto used by Mathematicians — reduce it to a mere assertion, the weakness of which becomes more apparent from the irresistible truths illustrated by that branch of science, Geometry, from which some of the so-called proofs of recognised Mathematicians are assumed to be derived, and, consequently, cannot be admitted, on the ground of their antiquity, or dogmatical teaching.

The following are the published works of the Author:

In 1861—"The Quadrature of the Circle," 226 pages.
1863—"Nut to Crack," 70 pages.
1865—"The Quadrature of the Circle," &c., 101 pages.
1866—"The British Association in Jeopardy," &c., 94 pages.
1867—"Letter to the Duke of Buccleuch," 74 pages.
1869—"Geometry of the Circle," 408 pages.
1870—"Curiosities of Mathematics," 2nd edition, 92 pages.
1870—"The Ratio of Diameter to Circumference in a Circle demonstrated," 470 pages.

In 1871—"Cyclometry and Circle-Squaring in a Nutshell," 44 pages.

1871—"Why is Euclid Unsuitable," &c., 52 pages.

1872—"The Quadriture and Geometry of the Circle Demonstrated," 282 pages.

The Author only completed the revision of this work for the printer about a week before his death.

Actuated by strong convictions of the earnestness of the Author's search for truth, and entertaining feelings of great respect for the memory of his deceased friend, the writer has requested permission of the executors to publish the present work, and they having kindly acceded, he offers it to the public in the hope that it may induce its readers to study Geometry from a new point of view, and so lead them to see the harmony which exists between Geometry and Mathematics, when the latter is properly applied to the former. The Author has shown this harmony by numerous diagrams and illustrations throughout his works, and he brings this hitherto abstruse question within the range of any thoughtful and careful reader, even though he may not have had a Mathematical training.

In conclusion, the writer would urge each reader to judge for himself as to the soundness of the Author's "arguments, reasonings, and conclusions," being convinced, that the maxim "*Magna est veritas et prævalebit*" will fairly apply.

THE EDITOR.

PREFACE.

THE following correspondence, in which I have proved that $\frac{24}{25}$ (circumference) in every circle, is exactly equal in arithmetical value to the perimeter of an inscribed regular hexagon, was in print some years ago, but has never yet been made public. The reader will find the reason in the closing Letters of the Correspondence, pages 248 to 257. My correspondent never attempted to refute my demonstrations, but contented himself by indirectly asserting that I must shew π to be a determinate quantity by *a priori* reasoning, *i.e.*, without reference to its arithmetical value. (See Mr. R.'s Papers of 8th and 16th April, 1868, and my Foot note, page 152.)

One of my correspondents, the Rev. Wm. Allen Whitworth, was an Honorary Secretary of Section A of the British Association at its meeting in Liverpool, in 1870. I never had but one conversation with that gentleman. It occurred in the reception room of the Association, in 1870, and the subject was the value of π. On that

occasion he denied the possibility of my proving the value of π to be $3\frac{1}{8}$, and charged me with assuming that value without a shadow of proof.

Such are fair examples of the treatment I have met with at the hands of our recognised Mathematicians.

Last month my friend "Geometricus" received a communication from a Mathematician to which it is not necessary to refer further than to say, that it led me to address the following letter to the Rev. Wm. Allen Whitworth :—

BARKELEY HOUSE, SEAFORTH,
16th March, 1872.

MY DEAR SIR,

Do not suppose I wish to drag you into a controversy, as I can assure you I have no such intention. My object is to shew *you*, that the arithmetical value of the symbol π is $3\frac{1}{8}$, and cannot be either greater or less; and throw upon you the *onus probandi* of refuting this conclusion. In this I ought to have no difficulty, since I know that you are a Scholar, a Mathematician, and a Christian Minister.

It is self-evident that every circle has a diameter, a circumference, and an area; and it would be very absurd to suppose that we cannot hypothetically assume an arithmetical value of the circumference to find the diameter and area, and, at same time, maintain that we can hypothetically assume the diameter of a circle to find the circumference and area.

The following things are admitted by all Mathematicians: First—Six times radius = the perimeter of a regular-inscribed hexagon in every circle. Second—The radius of a circle of diameter unity = $\cdot 5$.

Well, then, no Mathematician in the world can make the arithmetical value of π either *greater* or *less* than $3\frac{1}{8}$, without making the perimeter of a regular-inscribed hexagon to a circle of diameter unity either *greater* or *less* than 3; and the radius of a circle of diameter unity either greater or less than ·5, either of which suppositions is absurd.

From the foregoing facts we learn that, $\frac{24}{25}\left(\frac{\text{circumference}}{6}\right)$ = radius in every circle.

Now, Sir, if I am wrong, *you* as a scholar and a Mathematician can readily prove it; and if I am right, as a Christian Minister and a gentleman, you will frankly admit my conclusion to be irrefragable.

Waiting your reply to this communication,

I remain, my dear Sir,

Faithfully yours,

JAMES SMITH.

The Rev. Wm. Allen Whitworth, Liverpool.

I received the following answer:—

CHRIST CHURCH, LIVERPOOL,
March 18*th*, 1872.

MY DEAR SIR,

I acknowledge the receipt of your letter. But I have not time to enter into a correspondence with you on the subject of π.

Yours very truly,

W. ALLEN WHITWORTH.

James Smith, Esq.

Whenever I have forced opponents into a difficulty, they have almost invariably attempted to escape it by pleading want of time to enter into a correspondence;

the day, however, is not far distant when it will be universally admitted, that my conclusion, viz., "$\frac{24}{25}$ (circumference) in every circle, is exactly equal to the perimeter of an inscribed regular hexagon," never has been refuted, and never can be; and it, therefore, follows, *of necessity*, that 1 to $3\frac{1}{8}$ is the true and *exact* ratio of diameter to circumference in every circle.

JAMES SMITH.

EUCLID AT FAULT.

BARKELEY HOUSE, SEAFORTH,
1st October, 1868.

To JOSEPH DALTON HOOKER, ESQ., F.R.S., D.C.L., &c.

SIR,

I am a very old Life Member of "*The British Association for the Advancement of Science*," and have reason to believe that I am better known, than respected, by the leading Members of the Mathematical and Physical Section. The Astronomer Royal, in his opening address to that Section, at the thirty-first Meeting of the Association, held in Manchester, in 1861, observed :—"*It was known to those present that great ingenuity had been employed upon certain abstract propositions of Mathematics which had been rejected by the learned in all ages, such as finding the length of the circle, and perpetual motion. In the best academies of Europe, it was established as a rule that subjects of that kind should not be admitted, and it was desirable that such com-*

munications should not be made to the Section, as they were a mere loss of time."* These remarks arose out of a small pamphlet I distributed among the mathematical Members at that Meeting, a copy of which I had taken care to put the Astronomer Royal in possession of, previous to giving his opening address.

Notwithstanding the rules adopted "in the best academies of Europe," men—call them learned, or call them unlearned—have not been prevented from *spending* their time—*wasting* it the Astronomer Royal would say—on such subjects as "*The Quadrature and Rectification of the Circle;*" and I am not ashamed to confess that I am among the number. My labours have led to the discovery of the remarkable fact that, *Euclid is at fault* in one of his most important theorems; that is to say, that the eighth proposition of the sixth book of Euclid is not of general and universal application, and is therefore *not true* under all circumstances; and consequently, is inconsistent with the forty-seventh proposition of the first book. The proof of this fact is so plain and simple, as to be within the capacity of any man possessed of the most moderate geometrical and mathematical attainments; nay, I might say within the capacity of any first class school-boy; and to you, Sir, my demonstration will be as palpable, as that the square of 3 is 9.

I must now tell you how I was led to make this important discovery. I have for years attended regularly the Meetings of the British Association, but, not being allowed—by the rules of that body—to read a Paper in the Mathematical and Physical Section on "*The*

* Transactions of the British Association for 1861. Notices and Abstracts of Miscellaneous Communications to the Sections. Page 2.

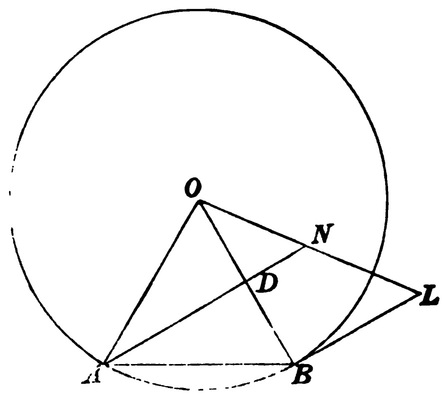
Quadrature of the Circle," I have, from time to time, brought out pamphlets on the subject, and these I have freely distributed among the mathematical Members of the Association. At the last Meeting, in Dundee, I distributed one. At the time I was writing that pamphlet, a Mr. and Mrs. S——, from Dumfriesshire, were on a visit to their son, resident in Liverpool; and being old friends of my wife's family, came out to Seaforth and made a call upon us. I happened to mention the fact to Mr. S—— that I was engaged in writing a Letter to His Grace the Duke of Buccleuch, the President Elect of the British Association, and told him the subject of it; when Mr. S—— informed me that in early life, he had himself been a good Mathematician, and still took a deep interest in Mathematics. This led us into a conversation on the subject of *Squaring the Circle*, which resulted in my presenting him with several of my pamphlets. He then told me that his brother was an excellent Mathematician, and a man of leisure; and, that he had a relative residing in the immediate neighbourhood of his brother, who was a first-class Mathematician, and that he should send the pamphlets to them, and induce those gentlemen to give them their careful attention, which his own health and business engagements would not admit of *his* doing. As soon as my Letter to the Duke of Buccleuch was published, I sent Mr. S—— copies, and in December last I received a communication from him, which led to a correspondence that would make a large volume, in which his relative, whom I may call Mr. R——, played the part of my chief opponent. Only some two or three communications passed between me and my friend's brother. Mr. S—— played the part, as

it were, of a medium, or I might say referee; that is to say, my letters were addressed to him, and after perusal, forwarded to his relative, and Mr. R——'s communications came to me through my friend, who really acted as referee, inasmuch as he kept Mr. R—— and me, within the legitimate bounds of controversy.

In the course of this correspondence, I think I extracted from Mr. R—— every conceivable objection he could advance against the truth of the theory, that 8 circumferences = 25 diameters, in every circle; which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π, and $\frac{1}{3{\cdot}125}$ the true expression of the ratio between diameter and circumference in every circle. I pointed out to Mr. R—— that in attempting to find the value of π, by multilateral sided inscribed polygons to a circle, whether we make an inscribed equilateral triangle to the circle, or an inscribed square to the circle, our starting point, the ratio of chord to arc in every successive polygon, is a varying ratio; and I shewed him that the reason is plain enough. It follows from the fact, that the sides of every successive polygon are convergent and divergent lines from the sides of those that precede them; and consequently, that we can never, by these processes, arrive at the value of π, or the true ratio of diameter to circumference in a circle: nor, can we arrive at them, by any other process, in which we attempt (directly) to measure a curved line by means of straight lines. Hence, the inapplicability of the forty-seventh proposition of the first book of Euclid, to measure directly the circumference of a circle. I answered every objection started by Mr. R——, still he was not convinced; and it then occurred

to me, that nothing short of proving the forty-seventh proposition of the first book, inconsistent with some other theorem of Euclid, would ever convince a recognised Mathematician that, the arithmetical value of π is a finite and determinate quantity. But, who ever thought of questioning Euclid? Professor de Morgan never went further than attempt to prove Euclid illogical* But, it never entered into *his mathematical philosophy*, to dream of proving Euclid *positively* at fault. How then was it likely that I should ever think of doing so?

Towards the end of April I was called away to Scotland, and on my return home spent a few days at Windermere, and it was during my stay there that I made the important discovery, that *Euclid is at fault.* The morning of the 2nd May was very wet at Windermere, and it occurred to me—as I could not leave the Hotel—that I could not better pass the time than by writing a letter to Mr. S—— enclosing a diagram, represented by the geometrical figure in the margin, in which the angle A and the sides O B and O L in the triangles O A B and O B L, are bisected by the line A N. This I intended as introductory to a



* Notes and Queries, 3rd S. VI. August 27, 1864. P. 161. Had the learned Professor asserted that the 18th and 19th Propositions of Euclid's third book are superfluous—what is proved by these propositions being established by the 16th proposition—I should have agreed with him.

succession of diagrams, explanatory and demonstrative of the important discovery, that, *the eighth proposition of the sixth book of Euclid is inconsistent with the forty-seventh proposition of the first book, and that it is the former, not the latter, that is at fault.*

It subsequently occurred to me, that if Euclid could be *at fault* in one Thorem, he might be *at fault* in others, and upon further examination I discovered, that *the twelfth and thirteenth propositions of the second book of Euclid, are also inconsistent with the forty-seventh proposition of the first book, and again, that it is the former, not the latter, that is at fault.*

You will observe that, so far, is word for word the same as the first part of my Letter to you of the 11th July, 1868. *(See Appendix A.)* The explanation is this:—I had resolved to publish so much of the Correspondence with Mr. S—— as I thought necessary to shew the train of reasoning by which a mathematician imagines he can refute the theory that 8 circumferences = 25 diameters in every circle, and by which I was led to the discovery that *Euclid is at fault* in the Theorem: Prop. 8, Book 6. Mr. R—— was so far a party to my publishing, that he revised his Papers from the one dated 19th February to that of the 1st April, and as so revised they appear in the following pages. I had returned him his own manuscripts for the purpose. I had about two hundred pages in type when I made the discovery that Euclid is at fault in two other Theorems. It was partly from this circumstance, that I was induced to lay the work aside for a time, and throw off my Letter to you of the 11th July, 1868, in writing which, I found I could not do better than adopt, as the introduction, the first six or

seven pages of what was printed off of my intended work. On again taking up my original intention, I found I must either commence, *de novo*, or merely make such alterations as were necessary to make the pages work in. I adopted the latter alternative, hence the repetition referred to.

I have made a selection from this Correspondence, and have only introduced so much of it as I thought absolutely necessary for the purpose I had in view; that is, of proving the true ratio of diameter to circumference in a circle, and demonstrating *Euclid to be at fault.* I have therefore omitted all my Letters to Mr. S—— previous to the 15th February, and we had exchanged many communications at that period. Some of these and Mr. R——'s rejoinders, I regret, I am obliged to exclude, to keep the work within reasonable limits.

CORRESPONDENCE.

MR. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
15th February, 1868.

MY DEAR SIR,

Your favour of yesterday is to hand, enclosing Mr. R——'s rejoinder to my letter of the 10th instant.

Mr. R—— is still unconvinced, that $\frac{24}{25}$ (circumference) = perimeter of a regular inscribed hexagon to a circle, and with reference to this he observes:—"*If Mr. Smith*

knows that it is true, he must have arrived at it by some process of reasoning. Let him reveal that process: let him take us into his confidence, and relate the whole steps of the process by which he discovered this fact." I can assure Mr. R—— that I have no wish to conceal, nor will I for a moment hesitate to reveal, the process of reasoning by which I arrive at that conclusion.

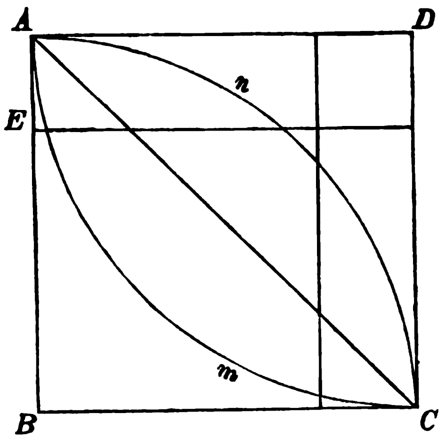
Take the anaolgy or proportion, $a : b :: b : c$. If $a = \frac{\pi}{4}$, and $b = 1$, then, $c =$ diameter of a circle of which the circumference $= 4$, whatever be the value of π. This, if not self-evident, is *axiomatic;* and of this fact Mr. R—— may readily convince himself, by means of any hypothetical value of π. Discovering this fact, it was at once obvious enough to me, that if the orthodox value of π could be true, no definite ratio between the diameter and circumference of a circle could possibly exist. But I found, *on the* THEORY, that 8 circumferences of a circle $= 25$ diameters, which makes $\frac{25}{8} = 3{\cdot}125$ the value of π; c is a finite quantity $= 1{\cdot}28$. But I also found that on this *theory*, $1{\cdot}28$ not only represents the diameter, but also represents the area of a circle of circumference $= 4$. Hence, 6 (radius) $= 6 \times {\cdot}64 = 3{\cdot}84 =$ perimeter of a regular inscribed hexagon to a circle of diameter $1{\cdot}28$. *I knew* that (circumference $\times$ semi-radius) $=$ area in every circle, and so found that $c \times s\ r = 4 \times \frac{1{\cdot}28}{4} = 4 \times {\cdot}32 = 1{\cdot}28 =$ area of a circle of circumference $= 4$. Now, since no other arithmetical symbols but $3{\cdot}125$ multiplied by $1{\cdot}28$ will produce 4;* it follows of necessity, that $3{\cdot}125$ must be the true value of π: and

* I should have added, *and make* $3{\cdot}84$ *the perimeter of an inscribed regular hexagon;* or, after the word symbols, the words, *within admissible limits.* Had I done so, Mr. R—— and I would have been spared much time and trouble.

$\frac{24}{25}$ (circumference) $= \frac{24}{25}(4) = 3\cdot84$ = perimeter of a regular inscribed hexagon to a circle, when circumference $= 4$.

Well then, divide the diameter into two parts, A B and B C, so that these two parts shall be in the ratio of 7 to 1. Then: $\frac{1}{8}(1\cdot28) = \cdot16 =$ B C; and, 7 (B C) $= \frac{7}{8}$ (A B + B C) $= 1\cdot12 =$ A B; therefore, A B^2 + B C^2 $= 1\cdot12^2 + \cdot16^2 = 1\cdot2544 + \cdot0256 = 1\cdot28$ = area of the circle. But, A B^2 — B C^2 $= 1\cdot2544 - \cdot0256 = 1\cdot2288 = 6$ (radius $\times$ semi-radius) $= 6\ (\cdot64 \times \cdot32) = 6 \times \cdot2048 = 1\cdot2288$ = area of a regular inscribed dodecagon to the circle; therefore, $\frac{\pi}{3}(1\cdot2288) = \frac{3\cdot125 \times 1\cdot2288}{3} = \frac{3\cdot84}{3} = 1\cdot28$ = area of the circle. Hence: $1\cdot2288 : 1\cdot28 :: 3\cdot84 : 4$: that is, area of dodecagon : area of circle : : perimeter of hexagon : circumference of circle.

Again: if the diameter of a circle $= 4$, the circumference and area are represented by the same arithmetical symbols, whatever be the value of π. Of this fact Mr. R—— may readily convince himself, by making the calculation on any hypothetical value of π. (This fact is also *axiomatic.*) But, on the *theory* that 8 circumference of a circle $= 25$ diameters; area of the circle $= 12\cdot5$, and area of a circumscribing square $= 16$, when the diameter of the circle $= 4$. Hence: when in the analogy $a : b :: b : c$, $a = 12\cdot5$ and $b = 16$, then, $c = 20\cdot48$; therefore, $\sqrt{a \times c} = \sqrt{(12\cdot5 \times 20\cdot48)} = \sqrt{256} = 16 = b$. But, $\frac{b^2}{a} = \frac{256}{12\cdot5} = 12\cdot8 = 10\ (1\cdot28)$ = ten times the value of c, when in the analogy $a : b :: b : c$, $a = \frac{\pi}{4}$, and $b = 1$.



In this geometrical figure let (BE + EA,) be a binomial, of which the two terms B E and E A are in the ratio of 3 to 1, by construction. Then: $BE^2 + 2(BE \cdot EA) + EA^2 = AB^2 = 16$; when $BE = 3$, and $EA = 1$. The Quadrants A *n* C B and D C *m* A are each equal to the area of an inscribed circle to the square A B C D. Now, let *a* denote the area of the square A B C D, let *b* denote the area of the square on E B, and let *c* denote the area of the inscribed quadrants. Then: On the *theory* that 8 circumferences of a circle = 25 diameters, $2\,(c) - a = b$. Hence: the area contained by the two arcs of the quadrants, is exactly equal to the area of the square on E B. Now, the square A B C D is divided by the diagonal A C into two equal right-angled isosceles triangles. Let *d* denote the area of one of these triangles. Then: $c - d$ = half the area contained by the arcs of the quadrants. Hence: $12{\cdot}5\,(AE^2) - d = \frac{1}{2}\,(EB)$, and this equation = the area contained by the diagonal of the square and one of the arcs of the quadrants. But, d + half the area contained by the arcs of the quadrants = $12{\cdot}5\,(AE^2)$, and the quadrants are indisputably equal in area to an inscribed circle to the square A B C D. Hence: $12\frac{1}{2}$ times the

area of a square on the semi-radius = area in every circle.

Take the following proof:—Let (B F + F A) be a binomial, of which the two terms B F and F A are in the ratio of 7 to 1, and the sum of the two terms = 4. Then: $BF = \frac{7}{8}(BF + FA) = \frac{7}{8}(AB) = 3{\cdot}5$, and $FA = \frac{1}{8}(AB) = {\cdot}5$; therefore, $BF^2 + FA^2 = 3{\cdot}5^2 + {\cdot}5^2 = 12{\cdot}25 + {\cdot}25 = 12{\cdot}5$ = area of a circle of diameter 4; and $BF^2 - FA^2 = 12{\cdot}25 - {\cdot}25 = 12 = 6$ (radius × semi-radius) = area of a regular inscribed dodecagon to a circle of diameter = 4. Hence: $1{\cdot}28 : 1{\cdot}28^2 :: 12{\cdot}5 : 4^2$.

Thus, by a process of reasoning from indisputable data, I arrive at the conclusion, that if the diameter of a circle be divided into two parts, so that these two parts shall be in the ratio of 7 to 1; the sum of the squares of these two parts = area of the circle; and the difference of the squares of these two parts = area of a regular inscribed dodecagon to the circle, which makes $\frac{24}{25}$ (circumference) = perimeter of a regular inscribed hexagon, in every circle.

In my last communication, I promised to direct your attention to some of the beauties of geometry, and with this view had prepared a Letter of some length. I intend to finish that Letter and post it in a day or two; in the meantime I have thought it better to simply comply with Mr. R——'s suggestion in the shortest way possible. With kind regards,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
17th February, 1868.

MY DEAR SIR,

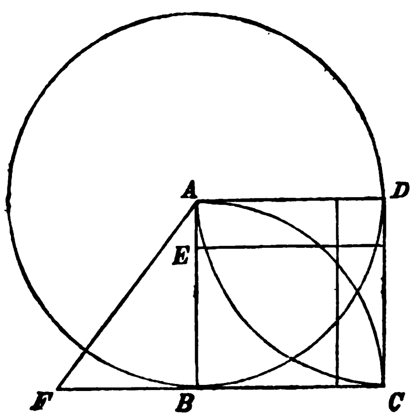
I am in receipt of your favour of the 15th inst., and regret to hear that Mrs. S—— has been so unwell. It is gratifying, however, to know that she is in a fair way of recovery.

I thank you very sincerely for your good wishes for my success in the controversy with your relatives, and you may rest assured, that they will not succeed "*in picking a hole in my coat.*" I perceive the difference between your brother and Mr. R——'s turn of mind; the former is more of the mechanical, the latter more of the mathematical. I shall take an opportunity of writing to your brother direct, on the subject of his diagram.

In my Letter of Saturday, I might have given other proofs, based on the geometrical figure in that Letter, of the truth of the *theory* that 8 circumferences = 25 diameters in every circle, which makes $\frac{24}{25}$ (circumference) = perimeter of inscribed hexagon. One in particular I should like Mr. R—— to have, and I take this opportunity of furnishing it. I cannot help thinking it will be convincing to that gentleman.

Make the following additions to the diagram:—

Produce C B to F, making B F = E B, and join A F, producing the right angled triangle A B F. With A as centre, and A B as interval, describe the circle.



Then:

$$CB + BF = AB + EB$$

$$\therefore AB - EB = AE$$

$$\therefore (AB + EB) + AE = 2\,AB.$$

Hence:

$$\pi\,(AB^2) = (AB + EB)^2 + AE^2.$$

But, $(AB + EB)^2 + AE^2 = AB^2 + BF^2 + AF^2$

$$\therefore AB^2 + BF^2 + AF^2 = \pi\,(AB^2).$$

Now, if not self-evident, it can readily be demonstrated to be ***axiomatic***, that when $AB = 4$, the area of the circle is exactly equal in numerical value to twice the circumference. We may adopt any hypothetical value of π, for the purpose of proof.

Again, $6\,(AB) = 6 \times 4 = 24$ = perimeter of a regular inscribed hexagon to the circle, when the radius $= 4$; and $\frac{25}{24}\,(24) = 2\,\pi\,(AB) = 25$ = circumference of the circle. If this stood alone, Mr. R—— would tell me, and very properly tell me, that I could only prove the equation, by assuming the value of π. But, it does not stand alone. For, $2\left\{\frac{25}{24}\,(24)\right\} = AB^2 + BF^2 + AF^2 = 50$ = area of the circle, when $AB = 4$. Again: $2\left\{\frac{25}{24}\,(24)\right\} = (AB + EB) + AE^2 = 50$ = area of the circle. Both these equations $= 2$ (circumference of the circle).

Hence: The equation, $(AB^2 + BF^2 + AF^2) = (AB + EB)^2 + AE^2 = 3{\cdot}125\,(AB^2)$ = area of the circle, and fixes $3{\cdot}125$ as the true arithmetical value of π; and it follows of necessity that the equation, $\{(AB + EB)^2 - AE^2\} = 6\left(AB \times \frac{AB}{2}\right) = 48$ = area of a regular inscribed dodecagon to the circle, and establishes beyond

the possibility of dispute or cavil, the truth of the major premiss of my proposition, when stated as a syllogism.

Mr. R—— will observe that, when AB the radius of the circle $= 4$, then, 2 (perimeter of an inscribed regular hexagon to the circle) $=$ area of an inscribed regular dodecagon. Hence: perimeter of hexagon : circumference of circle : : area of dodecagon : area of circle; $\frac{24}{25}$ (circumference) $=$ perimeter of hexagon; and, circumference of circle $= 5$ times the hypothenuse of the right-angled triangle ABF. Hence: We may put any value we please on the circumference of the circle, and demonstrate that it is equal to 5 (AF).

Will you kindly forward sheets 2 and 3 to Mr. R——? Hoping this may find Mrs. S—— better; with kind regards to her and all your circle,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. R——'s PAPER, *February* 19*th*, 1868.

In his Paper of the 15th, Mr. Smith comes nearer to his first principles. He says that it is *self-evident* that if in the analogy $a : b : b : c$; $a = \frac{\pi}{4}$, then $c =$ diameter of circle whose circumference $= 4$. This is neither self-evident nor *axiomatic;* yet is a true *consequence* of the definition of π and of the *demonstrable fact* that the diameters of circles are as their circumferences. Hence: $\frac{\pi}{4} : 1 : : 1 : c$, or $\pi : 1 : : 4 : c$; 4 being of course circumference to diameter c.

Mr. Smith proceeds: "If the orthodox value of π be true, there can be no definite ratio between the diameter and the circumference." This also is a matter of course. The orthodox value of π is an interminable decimal. But on the theory that $\pi = 3\frac{1}{8}$, he found that c is a determinate quantity and also represents the area of the circle, whose circumference $= 4$. And he concludes: "Now since no other arithmetical symbols but $3\frac{1}{8}$ and $1\frac{7}{25}$ will produce 4, it follows of necessity that $3\frac{1}{8}$ must be the true value of π."

Here then is Mr. Smith's whole process of reasoning, the first principles and the deduction. There are three elements in it. First: The assumption that the ratio π is a *determinate* number. Second: The assertion that no divisor but this will give a determinate quotient, *i.e.*, no number between the limits of 3 and 4 will thus divide 4. And, Third: That this quotient, being diameter (if $\pi = 3\frac{1}{8}$), also represents the area.

I am not aware that Mr. Smith means to make *all these three* essential to his process. I refer to the third. If he counts *it* essential, so much the worse for the process. WHATEVER the diameter may be, that diameter will $=$ area of circle whose circumference $= 4$. For diameter $= \frac{4}{\pi}$; and $\frac{rad}{2} = \frac{1}{\pi}$, and area $= 4 \times \frac{1}{\pi} = \frac{4}{\pi}$, the expression for diameter. But this is true of no other circle but the one whose circumference $= 4$. For if circumference $= m$, then diameter $= \frac{m}{\pi}$; and semi-radius $= \frac{m}{4\pi}$; and, since area $=$ circumference $\times$ semi-radius, the area here $= m \times \frac{m}{4\pi} = \frac{m^2}{4\pi}$. Now, this expression for the area, $\left(\frac{m^2}{4\pi}\right)$ can $= \frac{m}{\pi}$ ONLY when m, the circumference $= 4$.

But the other two elements must be essential, and $3\frac{1}{8}$ *is not proved* $= \pi$, unless (first) it be true that this ratio is determinate; and unless (second) this $3\frac{1}{8}$ be the only number

that will divide 4 *(or any number representing a circumference)* and give a determinate quotient.

Now, in the first place, what right have we to assume that this ratio is determinate? Is this self-evident? Is it a truth of an *axiomatic* character, involved in the very conception of a circle, as much as the truth that a whole is greater than its part, is involved in our conception of a whole and its part? Am I to understand that Mr. Smith says this: that every sane mind capable of understanding the definition of the circle, and of forming a conception of that figure, must see and admit without any proof, that the $\frac{\textit{circumference}}{\textit{diameter}}$ must give a determinate quotient? If he does not say this, then he has not yet given us his whole process; for he neglects to prove this essential element in his reasoning. I hold, until *this is proved*, that the question whether this ratio is determinate or not must wait for an answer, until we first discover its true arithmetical value.

Now if Mr. Smith can prove this, the problem is narrowed; but there is still another element; and I have to say, in the second place, that $3\frac{1}{8}$ and $1\frac{7}{25}$ are *not the only* arithmetical symbols whose product $= 4$. Here are 5 to add to Mr. Smith's one, making *at least* 6. There may be more within the limits; but 6 is a large number in such a case as this; and it puts a gulf between us and π, even if it (π) must be a determinate quantity. Here they are: $3\frac{1}{3} \times 1\frac{1}{5} = 3\frac{1}{5} \times 1\frac{1}{4} = 3\frac{4}{7} \times 1\frac{3}{25} = 3\frac{5}{9} \times 1\frac{1}{8} = 3\frac{7}{11} \times 1\frac{1}{10} = 3\frac{1}{8} \times 1\frac{7}{25} = 4$. And in all these cases, if the greater number $= \pi$, (and all are within the limits of 3 and 4,) the less number $=$ diameter, and also $=$ area, and I may well ask the *ground* or *grounds* on which Mr. Smith selects $3\frac{1}{8}$ out of these six numbers, which all fulfil the conditions he lays down?

In the third place, I said that if Mr. Smith counts the third element essential to his process, so much the worse for his process. I have shown that when the circumference $= 4$, the *diameter*, whatever its value, will represent the area; but that this is true of *no other circle*, but the one whose circumference

= 4. Mr. Smith's reasoning is therefore fallacious, because he makes (if he makes) what is true of only one particular circle, the one whose circumference = 4, an essential element in a process for discovering the most general of all the properties of the circle.

But I am willing to eliminate this element from the process, and leave it to Mr. Smith to vindicate the other two. If he shows them *both* to be *true*, he gains the day. But he will admit that both must be true, or his value of π is still unproved, and more than dubious.

When I asked a proof of the major of the Buccleuch demonstration, I was presented with the assertion that $\frac{24}{25}$ circumference = perimeter of an inscribed regular hexagon. When I asked proof of this, I am presented with an *assumption*, without proof; and an assertion that is not a fact. This is fatal to Mr. Smith's claim, unless he meet these two difficulties.

Mr. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
22nd February, 1868.

MY DEAR SIR,

I am in receipt of your favour of yesterday, enclosing Mr. R——'s last Paper, for which I am much obliged. You observe:—"*You* (that's me) *will be thinking you have caught a Tartar in my relative, Mr. R——. At it again, my boys.*" These remarks of yours remind me of pugulistic phraseology, and in every pugulistic contest there is a referee. Now, my dear Sir, you are the self-constituted referee in the mathematical contest between

me and your relative Mr. R——; and as I *know* that at your hands I shall get fair play, I have no doubt, as to the issue of the contest.

Well, then, in one part of his Paper, Mr. R—— charges me with having *asserted*, that the number 4 cannot be divided by any arithmetical quantity but $3\frac{1}{8}$, between the limits of 3 and 4, and give a determinate quotient. When, or where, have I ever made such an *assertion?* I beg to tell Mr. R——, that I know as well as he does, that $\frac{4}{3\frac{1}{8}}$ is a determinate quotient, and exactly equal to 1·25; or in other words, exactly equal to the radius of a circle, + one-fourth part of radius, when diameter = 2.

In another part of his Paper, Mr. R—— says:—"$3\frac{1}{8}$ *and* $1\frac{7}{25}$ *are not the only arithmetical symbols whose product is* 4. *Here are* 5 *to add to Mr. Smith's* 1, *making at least* 6. *There may be more within the limits, but* 6 *is a large number in such a case as this, and it puts a gulf between us and* π, *even if it* (π) *must be a determinate quantity. Here they are:* $3\frac{1}{3} \times 1\frac{1}{5} = 3\frac{1}{5} \times 1\frac{1}{4} = 3\frac{4}{7} \times 1\frac{3}{25} = 3\frac{5}{9} \times 1\frac{1}{8} = 3\frac{7}{11} \times 1\frac{1}{10} = 3\frac{1}{8} \times 1\frac{7}{25} = 4$; *and in all these cases, if the greater number* = π *(and all these are within the limits of* 3 *and* 4), *the less number* = *diameter, and also* = *area*." Now, I know quite as well as Mr. R——

That: $3\frac{1}{3} \times 1\frac{1}{5} = \frac{10}{3} \times \frac{6}{5} = \frac{60}{15} = 4.$

$3\frac{1}{5} \times 1\frac{1}{4} = \frac{16}{5} \times \frac{5}{4} = \frac{80}{20} = 4.$

$3\frac{4}{7} \times 1\frac{3}{25} = \frac{25}{7} \times \frac{28}{25} = \frac{700}{175} = 4.$

$3\frac{5}{9} \times 1\frac{1}{8} = \frac{32}{9} \times \frac{9}{8} = \frac{288}{72} = 4.$

$3\frac{7}{11} \times 1\frac{1}{10} = \frac{40}{11} \times \frac{11}{10} = \frac{440}{110} = 4.$

and, $3\frac{1}{8} \times 1\frac{7}{25} = \frac{25}{8} \times \frac{32}{25} = \frac{800}{200} = 4.$

The second and sixth of these equations, are expressible with arithmetical exactness, both fractionally and decimally. Will Mr. R—— venture to tell me, that he can give the other 4 in decimal expression with arithmetical exactness? I trow not!

Mr. R—— is a firm believer in Orthodoxy, which makes π an interminable decimal $= 3{\cdot}14159265...$, said to be true to these eight places of decimals, and is within the limits of $3\frac{1}{8}$ and $3\frac{1}{5}$. Now, suppose me to have said to Mr. R——: Let it be admitted that π cannot be less than $3\frac{1}{8}$, or greater than $3\frac{1}{5}$, would Mr. R—— have hesitated to make the admission? I hardly think he would! He could only have done so at the expense of convicting himself of being an unfair and uncandid controversialist! Had I taken the *precaution* to extract this admission from Mr. R——, it would have narrowed the issue between us; and he would have seen that his arguments, founded on his 5 pairs of arithmetical symbols whose product is 4, are not worth a straw as against the truth of my theory.

It is now obvious to me, that Mr. R—— either *cannot*, or *will not*, grapple with the question at issue between us, by means of practical geometry; and I must take him up in another way, and deal with him upon his own principles!

Well, then, *there is a circle* of which the values of the diameter and area are expressed by the same arithmetical symbols; and the circumference of this circle = 4. Mr. R—— proves this fact! But, *there is another circle*, of which the values of the circumference and area are expressed by the same arithmetical symbols, and the diameter of this circle $= 3\frac{1}{8} \times 1\frac{7}{25} = 4$.

Now, on the theory that 8 circumferences of a circle

= 25 diameters, which makes $\frac{25}{8}$ = 3·125 the arithmetical value of π; the area of the former circle = 1·28, and the area of the latter = 12·5, The area of a circumscribing square to the former circle = 1·6384, and the area of a circumscribing square to the latter = 16. Hence: 1·28 : 1·6384 : : 12·5 : 16; therefore, the product of the means is equal to the product of the extremes, and proves that the areas of circles are to each other as the areas of their circumscribing squares.

I now know Mr. R——, perhaps better than he knows himself; and I may tell *you*, that he would catch at this paragraph and say: Nobody would think of disputing your conclusion, that the areas of circles are to each other as the areas of their circumscribing squares; it follows, from the circumferences of circles being to each other as their diameters. But, you assume the value of π, and your conclusion proves nothing as to the value of π! This, however, would be anything but a fair way of putting it. What I do assume is this: *I assume a* THEORY *which makes* π = 3·125; and the truth of this *theory* I have proved in many ways by practical geometry. Many of these proofs I have given to Mr. R——, and he has never attempted to grapple with any of them.

Now, by hypothesis, let $\pi = 3·1416$, a close approximation to its true value, on the orthodox theory. Then: $\frac{4}{\pi} = \frac{4}{3·1416} = 1·273236 \ldots =$ diameter of a circle approximately; and, $\pi\left(\frac{1·273236}{2}\right)^2 = 1·273236 =$ area, when the circumference of the circle $= 4$. Again: $2\,\pi\,(2) = 12·5664 =$ circumference of a circle; and, $\pi\,(2^2) = 12·5664 =$ area, when the diameter of the circle $= 4$.

Then: $1{\cdot}273236^2 = 1{\cdot}621129910696$. Hence: $1{\cdot}273236 : 1{\cdot}621129910696 :: 12{\cdot}5664 : 15{\cdot}9999928605$ . . . The product of the means approximates very closely to the product of the extremes, and I have no objection to accept this as a proof, that the areas of circles are to each other as the areas of their circumscribing squares. But, will Mr. R—— venture to tell me that, this proof is as *perfect* as that derived from $\pi = 3{\cdot}125$? Have I not told Mr. R—— more than once, that there are many things in connection with geometry and mathematics, which may be proved by means of any hypothetical value of π ? With $\pi = 3\frac{1}{5}$ we may demonstrate with arithmetical exactness, that the areas of circles are to each other as the areas of their circumscribing squares. But, this does not make $\pi = 3{\cdot}2$. I would ask Mr. R——, if he can find a value of π within the limits of $3\frac{1}{8}$ and $3\frac{1}{5}$, which he can express with arithmetical exactness, both fractionally and decimally ? I may frankly assure him, that if he can, he knows more than I know ; and if he furnish the proof, I shall not *hesitate* to admit it.

Well, then, $3{\cdot}14159265$ . . . and $3{\cdot}125$, cannot both be the true circumference of a circle of diameter unity ; and for the sake of argument, it may be granted that neither may be the true value of π. It now appears that, so far, I have failed to carry conviction to Mr. R——'s mind by means of practical geometry, of the *truth of the theory that 8 circumferences* $=$ *25 diameters in every circle ; which makes* $\frac{25}{8} = 3{\cdot}125$ *the true arithmetical value of* π *; and makes* $\frac{24}{25}$ *(circumference)* $=$ *perimeter of a regular inscribed hexagon to every circle.* I must now change my tactics, and since I find Mr. R——'s *mind* impervious to geometrical proof, try if I can find an entrance into it by means of mathematical demonstration.

Well, then, take the following example of continued proportion :—

$$a : b :: b : c :: c : d :: d : e.$$

Let $(a + \frac{1}{4}a) = b : (b + \frac{1}{4}b) = c : (c + \frac{1}{4}c) = d :$ *and* $(d + \frac{1}{4}d) = e.$

Then : If $a = 4$.

$$(a + \tfrac{1}{4}a) = 4 + 1 = 5 = b$$
$$(b + \tfrac{1}{4}b) = 5 + 1{\cdot}25 = 6{\cdot}25 = c$$
$$(c + \tfrac{1}{4}c) = 6{\cdot}25 + 1{\cdot}5625 = 7{\cdot}8125 = d.$$

and, $(d + \frac{1}{4}d) = 7{\cdot}8125 + 1{\cdot}953125 = 9{\cdot}765625 = e.$

Hence : $\sqrt{e} = \sqrt{9{\cdot}765625} = 3{\cdot}125 = \pi$, on the *theory* that 8 circumferences of a circle are exactly equal to 25 diameters.

Now, $\frac{e}{\frac{1}{4}\pi} = e \times 1\frac{7}{25}$, that is, $\frac{9{\cdot}765625}{{\cdot}78125} = 9{\cdot}765625 \times 1{\cdot}28$, and this equation $= 12{\cdot}5$, and is the area of a circumscribing square to a circle of which π^2 is the area. Hence: $\sqrt{12{\cdot}5}$ = diameter of a circle of which the area $= \pi^2$, and $12{\cdot}5$ is the area of a circle of which the diameter $= 4$. For : $\frac{1}{2}(\sqrt{12{\cdot}5}) = \sqrt{3{\cdot}125}$ = radius of a circle of diameter $\sqrt{12{\cdot}5}$; therefore, $\pi\,(r^2) = \pi\,(\sqrt{3{\cdot}125}^2) = 3{\cdot}125^2$ = area of the circle, when diameter $= \sqrt{12{\cdot}5}$: and, $\pi\,(2^2) = 3{\cdot}125 \times 4 = 12{\cdot}5$ = area of the cirle, when diameter $= 4$.

Now, let the diameter of a circle $= \sqrt{12{\cdot}5}$. Divide the diameter into two parts, A B and B C, in the ratio of 7 to 1. Then : (A B + B C) is a binomial, and the sum of the two terms, A B and B C = A C = $\sqrt{12{\cdot}5}$ = diameter of the circle. B C $= \frac{1}{8}(\sqrt{12{\cdot}5}) = \sqrt{\left(\frac{1^2}{8^2} \times 12{\cdot}5\right)} = \sqrt{(\frac{1}{64} \times 12{\cdot}5)} = \sqrt{({\cdot}015625 \times 12{\cdot}5)} = \sqrt{{\cdot}1953125}$: and, A B = 7 (B C) $= \sqrt{(7^2 \times {\cdot}1953125)} = \sqrt{(49 \times {\cdot}1953125)} = \sqrt{9{\cdot}5703125}$: therefore, A B^2 + B C^2 =

$9{\cdot}5703125 + {\cdot}1953125 = 9{\cdot}765625 = \pi^2 =$ area of the circle. But, $AB^2 - BC^2 = 6$ (radius $\times$ semi-radius), that is, $9{\cdot}5703125 - {\cdot}1953125 = 6\,(\sqrt{3{\cdot}125} \times \sqrt{{\cdot}78125})$ $= 6\,(\sqrt{2{\cdot}44140625}) = 6 \times 1{\cdot}5625 = 9{\cdot}375 =$ area of an inscribed regular dodecagon to the circle. Hence: Since the property of one circle is the property of all circles, it follows of necessity, that if the diameter of a circle be divided into two parts, in the ratio of 7 to 1, the sum of the squares of these two parts = area of the circle; and the difference of the squares of these two parts = area of an inscribed regular dodecagon to the circle; and makes $\frac{24}{25}$ (circumference) = perimeter of a regular inscribed hexagon in every circle.

In the foregoing example of continued proportion let $a = 1$.

Then:

$(a + \frac{1}{4}\,a) = 1 + {\cdot}25 = 1{\cdot}25 = b.$

$(b + \frac{1}{4}\,b) = 1{\cdot}25 + {\cdot}3125 = 1{\cdot}5625 = c = \frac{\pi}{2}.$

$(c + \frac{1}{4}\,c) = 1{\cdot}5625 + {\cdot}390625 = 1{\cdot}953125 = d.$

and, $(d + \frac{1}{4}d) = 1{\cdot}953125 + {\cdot}48828125 = 2{\cdot}44140625 = e = c^2.$

Again: Let $a = 1\frac{7}{25} = 1{\cdot}28.$

Then:

$(a + \frac{1}{4}\,a) = 1{\cdot}28 + {\cdot}32 = 1{\cdot}6 = b.$

$(b + \frac{1}{4}\,b) = 1{\cdot}6 + {\cdot}4 = 2 = c.$

$(c + \frac{1}{4}\,c) = 2 + {\cdot}5 = 2{\cdot}5 = d.$

and, $(d + \frac{1}{4}\,d) = 2{\cdot}5 + {\cdot}625 = 3{\cdot}125 = e = \pi.$

In all these examples, $\sqrt{a \times e} = \sqrt{b \times d}$, and this equation $= c$; therefore, c is a mean proportional between the extremes of a and e and b and d. Hence:

If a denote the area of a square, c = area of a circumscribing circle.

Now, my dear Sir, Mr. R—— may as well attempt to prove that the square of 7 = 50, as attempt to controvert these proofs of the truth of the *theory* that 8 circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3·125$ the true value of π, and makes $\frac{4}{25}$ (circumference) = radius, in every circle.

In the enclosed diagram (*See Diagram* I.), if the line C D = 4, C E = 5, C R = 7, D E and R L = 1, and C L = 8, by construction. Hence, the diameter of the circle Y = twice the diameter of the circle X. Now, if Mr. R—— knows anything of practical geometry, and chooses to apply his knowledge, he may readily convince himself that the area of the square E C G F = half the area of the circle Y, and twice the area of the circle X. Hence: $\frac{1}{8}$th part of the perimeter of any square = diameter of a circle containing half the area of the square: and $\frac{1}{8}$th part of the circumference of any circle = side of a square containing half the area of the circle. Indeed, Mr. R—— will discover that this geometrical figure is a perfect study, and abounds with geometrical truths, of which I cannot help thinking he is at this moment profoundly ignorant. But as matters stand I shall say no more about it at present.

In conclusion I beg to observe, Mr. R—— may, or may not, have been in possession of my Letter to you of the 17th inst., when he wrote his Paper; at any rate, he has not referred to it. With regard to that of the 15th inst., to which his paper professes to be a rejoinder, I can hardly conceive it possible that he can have read beyond the middle of the second page. If he did read

Mr. James Smith *to* Mr. Jno. S——.

Barkeley House, Seaforth,
25th February, 1868.

My Dear Sir,

Your brother, some days ago, kindly sent me a diagram of yours, with the following quotations from your Letter to him.

"Mr. R—— has had another Letter from Mr. Smith, and has replied to him. As I have had the diagram herewith sent, sometime before, I humbly submit it also, for inspection; as it shews that, if the calculations are right, the result does not tally with Mr. Smith's theory."

"They are all founded on the 47th of the 1st Book of Euclid, but if so, they rest on a solid foundation, and I cannot see any objections to the use of it. If Mr. Smith can point out any, I will weigh them. They are just a series of applications of it, the one following on the other, in order to get the length of lines, without any reference to curves—only straight lines. Who can object to such a use of the proposition, that the square of the hypothenuse is equal to the squares of the other two sides of every right-angled triangle?"

Now, my dear Sir, I maintain, as firmly as you do, the "*solid foundation*" of that wonderful problem of Pythagoras, that the square of the hypothenuse is equal to the sum of the squares of the other two sides, in every right-angled triangle. But, I dispute your *assumption* that you "*get the length of lines without any reference to curves,*" by *your* application of the 47th of the first Book of Euclid, and I will tell you *why* I dispute your assumption.

that Letter through, he has chosen to treat my arguments, derived from a circle of diameter $= 4$, with silent contempt. Can this be called fair controversy? He had asked me to relate the whole steps of the process by which I arrive at the conclusion, that $\frac{24}{25}$ (circumference) $=$ perimeter of a regular inscribed hexagon in every circle. Was not *that* one step, and that a very important step, in the process by which I arrive at that conclusion? We may pervert Scripture by tampering with a text apart from its context; and so, Mr. R—— may pervert my arguments, if he choose to catch at an expression in a Letter, without proper regard to the context. This is exactly what Mr. R—— has done, with reference to my Letter of the 15th, and I think you will agree with me, that this is hardly treating me with the same candour and frankness that I have treated him. But never mind; I think he will be puzzled to *pick a hole* in this communication.

I was very unwell for several days last week, and did not get the Letter written to your brother, and indeed I begin to think it is hardly necessary. I presume he will see this, and if it does not convince, nothing will.

I am glad to hear that Mrs. S—— is somewhat better, though still weak, and with kind regards,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Construct a regular hexagon. Does not even the construction of a regular hexagon involve curves? But, admitting, for the sake of argument, that you can, and do, construct a regular hexagon, independently of a circle; when constructed, you will admit that you can find the centre of the hexagon. Then: If with one point of your compasses on the centre, and an interval equal to the radius of the hexagon, you describe a circle, will not the circumference of the circle touch all the angles of the hexagon? Hence: You cannot fail to perceive, that a circle is involved in the construction of all polygons of more than 4 sides. Euclid: Prop. 11, Book 4, shows us how to construct an equilateral and equiangular pentagon in a given circle; but, no such pentagon can be constructed, independently of a circle. *There is no rule of thumb*, in Geometry.

In the enclosed diagram, (*see Diagram* II.) the figure O A E B is a quadrant, of which O A, O E, O B, &c. are radii. O C B is an equilateral triangle, of which the sides are equal to the radius of the quadrant. Hence: C B is a side of a regular inscribed hexagon to a circle, of which O B and O C are radii. The chords C D and D B are sides of a regular dodecagon, and the chords C E, E D, D F, and F B are sides of a regular 24 sided polygon, to a circle of which O B and O C are radii.

Now, it would be very absurd if I were to say that the sum of the two chords C D and D B, or, the sum of the four chords C E, E D, D F, and F B, was not greater than the chord C B. Now, C B = radius of the quadrant; therefore, 2π (C B) = circumference of a circle of which O B and O C are radii. Well, then, by hypothesis, let the chord C E, a side of the 24 sided polygon, = 4

Then: The equation, $24\left\{\frac{\pi}{3}(\text{chord } CE)\right\} = 6\left\{\frac{\pi}{3}(\text{chord } CB)\right\}$ gives the circumference of the circle, whatever be the value of π. For example: By hypothesis, let $\pi = 3{\cdot}1416$. Then: $24\left\{\frac{\pi}{3}\cdot(\text{chord } CE)\right\} = 6\left\{\frac{\pi}{3}(\text{chord } CB)\right\}$, that is, $24\left(\frac{3{\cdot}1416 \times 4}{3}\right) = 6\left(\frac{3{\cdot}1416 \times 16}{3}\right) = 100{\cdot}5312 =$ circumference of a circle of which O A or O B is the radius, on the hypothesis that $\pi = 3{\cdot}1416$. What!—you may say—do you mean to tell me that the four chords, C E, E D, D F, and F B are together only equal to the chord C B? No! my dear Sir, I don't mean to tell you anything of the kind! I have already told you it would be very absurd to do so! But, will you tell me that 100·5312 is *not* the circumference of a circle, of which 4 times 4 is the radius, on the hypothesis that $\pi = 3{\cdot}1416$? This is a puzzle to Mathematicians, but is readily explainable, and I will give you the explanation before I conclude this epistle. Well, then, 100·5312 is the circumference of a circle of radius = 16, on the hypothesis that $\pi = 3{\cdot}1416$. Then: $\frac{100{\cdot}5312}{24} = 4{\cdot}1888 =$ arc subtending the chord C E. Hence: $24\left\{\frac{3}{\pi}(\text{arc } CE)\right\} = 6\,(CB)$, that is, $24\left(\frac{3 \times 4{\cdot}1888}{3{\cdot}1416}\right) = 6 \times 16 = 96 =$ perimeter of a regular inscribed hexagon to a circle, of which O B or A B = C B is the radius, and radius = 16. But, any other hypothetical value of π will produce the same result.

Again: If, by hypothesis, we make the chord C D = 8. Then: The equation, $12\left\{\frac{\pi}{3}(\text{chord } CD)\right\} = 6\left\{\frac{\pi}{3}(\text{chord } CB)\right\}$ gives the circumference of the circle as

before; and if we carry out the operation, the calculations work out to the perimeter of the hexagon.

If, by hypothesis, we make the value of $\pi = 3{\cdot}125$, the only difference is, that this hypothesis makes the circumference of the circle 100 instead of 100·5312; and of course, if we assume π to be less than 3·125, we should make circumference less than 100. But, whatever hypothesis we may adopt, if we work out the calculations, the final result is the same: we get the perimeter of the hexagon = 6 (radius).

Now, describe a circle, and inscribe a regular pentagon, and a regular hexagon. Euclid teaches us how to construct the pentagon, and the merest tyro in geometry knows how to construct the hexagon. Well, then, let the circumference of the circle = 360. I only fix upon 360, because, for all practical purposes, we *assume* 360 as the circumference of a circle. Divide it into 360 equal parts which we call degrees, and again sub-divide these into 60 equal parts which we call minutes. Now, it would be very absurd, if I were to say, that the perimeter of the pentagon was exactly equal to the perimeter of the hexagon. But, mark! $\frac{360}{5} = 72$. From this deduct $\frac{1}{25}$th part $= 2{\cdot}88$. Then: $72 - 2{\cdot}88 = 69{\cdot}12$. Again: $\frac{360}{6} = 60$. From this deduct $\frac{1}{25}$th part $= 2{\cdot}4$. Then: $60 - 2{\cdot}4 = 57{\cdot}6$. Hence: $5(69{\cdot}12) = 6(57{\cdot}6)$; therefore, this equation $= 345{\cdot}6$, and is equal to the perimeter of a regular inscribed hexagon to a circle of circumference 360. But, $\frac{4}{25}$ (circumference) $= \frac{3}{3{\cdot}125}$ (arc subtending a side of the hexagon), that is, $\frac{4}{25}(360) = \frac{3}{3{\cdot}125}(60)$; therefore, this equation $= 57{\cdot}6$, and is equal to radius of a

circle of circumference 360. Hence: $\frac{3 \cdot 125}{3}$ (345·6) = $6\left\{\frac{3 \cdot 125}{3}(57 \cdot 6)\right\}$; therefore, this equation = 360 = circumference of the circle; and since 3 is the perimeter of a regular inscribed hexagon to a circle of diameter unity = 1, it follows, that 3·125 is the true circumference of a circle of diameter 1.

Well, then, I must now give you an explanation of the apparent anomaly in geometry, to which I have referred, and which would appear to be a perfect puzzle to Mathematicians. If the circumference of a circle be divided into any number of equal arcs, and from one of these arcs $\frac{1}{25}$th part be deducted, and the remainder multiplied by the number of arcs into which the circumference of the circle is divided, the product is a constant quantity, and exactly equal to the perimeter of a regular hexagon inscribed in the circle. I have been in correspondence with many Mathematicians, but have never found *one* who dare dispute this fact; and yet, I have never met with *one* who has had the candour to admit it, and your relative, Mr. R——, is no exception.

Now, in the same circle, the perimeter of a regular inscribed hexagon : circumference of circle : : area of a regular inscribed dodecagon : area of circle; and I defy any Mathematician to find two inscribed polygons of a greater number of sides than the hexagon and dodecagon, to any circle that will give the same analogy or proportion, expressible with arithmetical exactness. Then: Surely, my dear Sir, you cannot fail to perceive that you can never prove, either what π is, or what π is not, by your *direct* application of the 47th of the 1st Book of Euclid.

Now, permit me to refer you to the diagram. You will observe, that the point C is an extremity of the three chords C B, C D, and C E. But, the chords C D and C E are divergent lines from the point C. Hence, the ratio of the chord C D to its subtending arc, is a different ratio from that of the chord C B to its subtending arc; and the ratio of the chord C E to its subtending arc, differs from both; therefore, the chord C E is a longer line in proportion to its subtending arc, than the chord C D to its subtending arc; and the chord C D is a longer line in proportion to its subtending arc, than the chord C B to its subtending arc. Now, if we went on multiplying the number of sides of a polygon to a circle of which O B is the radius, the ratio of chord to arc would be a varying ratio at every step. Hence, it is utterly impossible to prove either what π *is*, or what π *is not*, by a direct application of the 47th of the 1st Book of Euclid.

Now, my dear Sir, have the two following facts never occurred to you? First: To whatever extent we may double the number of sides of inscribed polygons to a circle, as we increase the perimeters of the polygons we increase the areas in like proportion, but can never arrive at a polygon equal in perimeter and area to its circumscribing circle. Second: It is conceivable that in the generating circle, circles might be inscribed of which the circumferences should be equal to the perimeter of the inscribed polygons. From these two facts we get, what appears to me, one of the strongest arguments against the *direct* application of the 47th of the 1st Book of Euclid in search of π, and to this I will now direct your attention.

Well, then, let the radius of a circle = 1. Then:

The perimeter of an inscribed regular hexagon = 6, and the area of the hexagon = 2·598075.* If to this be added the sum of the areas of the 12 right-angled triangles about the hexagon, which make up an inscribed dodecagon or 12-sided polygon to the circle = ·401925*, we obtain the area of the dodecagon = 6 (radius × semi-radius) = 3. Now, conceive a circle to be inscribed within a circle of radius = 1, so that the circumference of this circle shall be equal to the perimeter of a regular inscribed hexagon to the circle of radius 1 = 6. Then: If x denote the area of the circle, y the area of the dodecagon, and z the area of the circumscribing circle to the dodecagon, then, $3 : \pi :: x : y$; and $3^2 : \pi^2 :: x : z$, whatever be the value of π.

Now, $3\left(\frac{6}{6 \cdot 25}\right) = 3 \cdot 125 \left(\frac{6}{6 \cdot 25}\right)^2$, and this equation = 2·88, and is exactly equal to the area of a circle of circumference 6, on the *theory* that 8 circumferences of a circle = 25 diameters, which makes $\frac{25}{8}$ = 3·125, the value of π; and we cannot make the area of a circle of circumference 6 differ much from 2·88, by whatever value of π, *within admissible limits*, we make the calculation. Then: area of circle — area of hexagon = 2·88 — 2·598075 = ·401925 = area of the 12 right-angled triangles about the hexagon, which make up the dodecagon. Thus, area of circle of circumference = 6 + area of triangles = 2·88 + ·401925 = 3·281925. This is an

* 2·598075 is rather less, and ·401925 rather greater than the true value. Both are, in reality, incommensurable quantities, and I have only made them finite for convenience. This is one of those examples, of which we have many in Geometry, in which two incommensurables make one commensurable.

arithmetical quantity greatly in excess of the area of a circle of radius = 1, whether calculated on π 3·1416, or 3·125. But, if to the area of a circle equal in circumference to the perimeter of the dodecagon, we add the difference between the areas of 12 and 24-sided polygons, we obtain a smaller arithmetical quantity than 3·281925; and by continuing this process we may still further reduce this quantity at every step; but when we have extended the calculations to the exhausting point, we have still a quantity in excess of the area of a circle of radius 1. Hence: the 47th proposition of the first Book of Euclid is inapplicable *(directly)* to the measurement of a circle. It appears to me, that Mathematicians altogether lose sight of the important fact with regard to the question at issue, that a line of any given length will enclose a larger superficies if in the form of a circle, than it can be made to enclose in any other form whatever.

I shall be pleased to find that I have been successful in expressing my views, and have carried home conviction to your mind.

I remain, my dear Sir,

Yours very truly,

JAMES SMITH.

Jno. S——, Esq.

MR. R——'S PAPER, *26th February*, 1868.

In his Paper of the 15th, Mr. Smith revealed the whole process by which he had arrived at $\pi = 3\frac{1}{8}$. In my last I showed that this process proceeds on the *principle* that π *is*

determinate. To this I called Mr. Smith's attention, and requested a proof of a premiss so *fundamental* to his conclusion. If he cannot show satisfactorily that this premiss *is true*, what right has he to insist that I or any man admit his conclusion? Now, in his Letter of the 22nd, Mr. Smith takes no notice of this; says not a syllable about it; and what conclusion am I to draw from this reticence? Putting his reasoning as a syllogism, it stands thus:—.

The ratio between the circumference and the diameter is a determinate quantity. $3\frac{1}{8}$ is the only determinate quantity that will give a "finite" quotient with (4); therefore, $3\frac{1}{8}$ is the ratio in question. I hold that this is a fair and correct analysis of Mr. Smith's process of reasoning in search of π. He says: "If the orthodox value of π could be correct, no definite ratio between the diameter and circumference could exist." But if $3\frac{1}{8}$ be taken as this ratio, the diameter of a circle whose circumference $= 4$ is a finite quantity ($1\frac{7}{25}$). "Now since no other arithmetical symbols than $3\frac{1}{8} \times 1\frac{7}{25}$ will produce 4, it follows of necessity that $3\frac{1}{8} =$ true value of π." (Mr. Smith complains that I charge him with *asserting* that the number 4 cannot be divided by any number but $3\frac{1}{8}$ and give a determinate quotient; and asks where he ever made such an assertion. To say, that "$3\frac{1}{8} \times 1\frac{7}{25}$ *are the only arithmetical symbols that will produce* 4" is the *same* as to say, that $3\frac{1}{8}$ *is the only number that will divide* 4 *and give a determinate quotient.* To me these seem *identical* propositions. I cannot see how *Mr. Smith can question their identity.* If they are not identical, then I have misrepresented the proposition; but if Mr. Smith cannot show that these two propositions are not identical, in asserting the one he asserts the other; and it is perfectly legitimate for me to substitute the one form for the other. Is it fair in him to complain of this?)

Now, if the above syllogism be a correct exhibition or analysis of Mr. Smith's reasoning on the point, the major and the minor must both be true to warrant the conclusion. But in my last I disputed both these premisses. The major is not self-evident; therefore it must be proved. The minor I also

disputed, and showed that there were other numbers than $3\frac{1}{8}$ that met the conditions Mr. Smith laid down. In his reply Mr. Smith ought to have defended his major. Even were his *minor* true, the conclusion cannot stand on *it* alone. Can it? If Mr. Smith do not establish that major, obviously his process is invalid.

But he must dispose of my objections to the minor also, and he has not met these. He says that only *one* of my 5 numbers can be expressed by a terminable decimal, and assumes that this puts the others out of court. This is to introduce a new and arbitrary condition, and the necessity of this is not established by Mr. Smith. Are not $3\frac{4}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, $3\frac{1}{3}$ all as *determinate* as $3\frac{1}{5}$ and $3\frac{1}{8}$? How can this new condition be justified? Mathematics have always been called the *exactest* of sciences. But this way of introducing new conditions by mere assertion, makes *exact* treatment impossible. If the condition was *necessary* Mr. Smith should have stated it in his *process*. This condition necessitates another syllogism, which takes the following shape:—

The ratio between circumference and diameter must be expressed in a determinate decimal. $3\frac{1}{8}$ is the only number that obeys this condition; therefore, $3\frac{1}{8} = \pi$.

Now, let Mr. Smith justify the major. I dispute it, and ask proof. But, moreover, by Mr. Smith's own confession, $3\frac{1}{5}$ also meets the condition he lays down; and therefore $3\frac{1}{8}$ is *not* the only number, and the *minor* is not true. Is this reasoning? Mr. Smith complains that I have dealt unfairly with his arguments; but I am conscious of the very opposite feeling. Only I have subjected his reasoning to a rigorous analysis, and shown him where it fails. He cannot complain of this with any justice; seeing that he professes by *this reasoning to revolutionise* the science of mathematics; and holds that the greatest mathematical geniuses have been wrong in this so fundamental matter.

Now up to this point I have only shown that by this process, Mr. Smith has *not proved* that $3\frac{1}{8} = \pi$. I have not tried to

prove, nor have I *asserted* that π *is not* $= 3\frac{1}{8}$. He says that I am "a firm believer in the orthodox π." I may ask when I have said that? I have said that all Mr. Smith's attempts to convince me that $\pi = 3\frac{1}{8}$, serve to rather convince me that π is *not determinate*. I further avow my belief in the correctness of the *methods* by which Mathematicians have sought the value of it, I mean the purely geometrical one, and the other by the differential and integral calculus. But I am open to light if only it be genuine.

I now come to his complaint that I do not grapple with his proofs by *practical* geometry. Why, it was my grappling with one of them—that sent to the Duke of Buccleuch—that brought us to this pass. More than this I have declared my readiness to grapple with the *whole* of them. But it is far more to the point to go to the first principles with which Mr. Smith proceeds. In his last he says he *assumes the* THEORY that $\pi = 3\frac{1}{8}$, and proves it true in many ways by *practical geometry*. But is this consistent with the profession he made in a previous communication, to have revealed the whole process of reasoning by which he proved $\pi = 3\frac{1}{8}$? If a thing is *proved* by a *process of reasoning*, how can it be *assumed* as a THEORY? Mathematics *admit of no assumed theories*. Every thing must be proved, rigorously from the axioms or identical propositions, or no assent can be demanded.

But I further assert that if Mr. Smith means by *practical* geometry what every other person means, he can prove *nothing* by it. Practical geometry is merely the application to the construction of figures, or the doing of something with lines or figures, which *theoretical* geometry has already shown us the way to do. Practical geometry is to theoretical, what the rules of arithmetic are to the higher and far more interesting and recondite *analysis* by which they are derived. None of Mr. Smith's geometrics is a *proof* that π is $3\frac{1}{8}$. In every one there is the impassable hiatus.

Take the very last. On page 10, referring to the figure, Mr. Smith says "If Mr. R—— knows anything of practical

Geometry, and chooses to apply his knowledge, he may convince himself that square of E C, *i.e.*, ($\frac{5}{8}$ diameter)^2 = $\frac{1}{2}$ the area of circle Y." *(See Diagram I.) This is mere trifling with me.* It is to say that I can prove the very thing I ask Mr. Smith to prove; and if I cannot, or will not, that by implication I know nothing of geometry. If Mr. Smith *could prove* this himself, he would have done it long ago.

Now the sum of the whole matter is, that the *reasoning* by which Mr. Smith arrived at his π is fundamentally wrong, unless he has something in reserve. But as I asked this in my last, and nothing has come, I conclude that there is nothing behind. This process therefore is a failure. Second. The *assumed theory* has not been proved by practical geometry, and *cannot* be proved by it. It must be proved, if done, by *theoretical* geometry. I know enough of geometry to understand anything in the shape of proof that Mr. Smith can offer. Indeed my chief difficulty all along has been the confident way in which he has *asserted* the very thing that requires rigorous proof. I am ready to justify this complaint by analyzing the other modes with which Mr. Smith has favoured us.

The division of the diameter by 7 : 1 produces certain results. For instance, the difference of the squares of these parts = the inscribed dodecagon. But this has no connection whatever with π. The *real nexus* between this division and π is, that the *sum* of these squares = the *area* of the circle; and this Mr. Smith *asserts*, not *proves*. Were $3\frac{1}{8}$ = π, *then* the square = the circle would thus be constructed and exhibited. But that *does not prove* that $\pi = 3\frac{1}{8}$. Surely Mr. Smith will not say that it *does*.

Again, he ASSERTS that $\frac{24}{25}$ circumference = $6r$ or $3d$, the perimeter of a regular hexagon; and I see this assertion repeated in his Letter to Mr. Jno. S—— with the same confidence as before, and myself named as wanting the candour to admit it. This is simply power of *assertion*, not force of *reasoning*. If, *again*, π were = $3\frac{1}{8}$, this would be true; or if this could be shown to be true independently of π, it would follow that $\pi = 3\frac{1}{8}$. But

to use the former to prove the latter, and then the latter to prove the former, is simply astounding; it is not reasoning at all.

Glancing at the *mathematical demonstration* by continued proportion, I have only to say that it *demonstrates* nothing but that 9·765625 = the 5 terms in a series of which the first term is 4 and the common ratio $\frac{5}{4}$. To offer this as a PROOF that $3\frac{1}{8} = \pi$, is playing with me.

Mr. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
4th March, 1868.

MY DEAR SIR,

Yesterday morning's post brought me your favour of the 2nd inst., enclosing Mr. R——'s rejoinder to my Letter of the 22nd inst. It would be an act of folly on my part to attempt to analyze such a paper, unless, and until I *know*, that Mr. R—— and I are agreed as to the following propositions :—

First: $\pi\ (r^2)$ = area in every circle, whatever be the value of π.

Second: $\frac{\pi}{3}$ (perimeter of a regular hexagon) = circumference of a circumscribing circle, whatever be the value of π.

Third: $\frac{\pi}{3}$ (area of a regular dodecagon) = area of a circumscribing circle, whatever be the value of π.

Fourth: The natural sines of angles in similar right-angled triangles must necessarily be expressed by the same arithmetical symbols.

Fifth: The area of an inscribed square to any circle is equal to half the area of a circumscribing square to the same circle.

Sixth: The area of an inscribed circle to any square is equal to half the area of a circumscribing circle to the same square.

Seventh: The circumference of a circle is equal in numerical value to half the area, when the diameter $= 8$, whatever be the value of π.

Eighth: The circumference and area of a circle are equal in numerical value, when the diameter $= 4$, whatever be the value of π.

I hardly need observe, that the diameter and area of a circle are equal in numerical value, when the circumference $= 4$, since Mr. R—— himself has demonstrated this fact.

Well, then, let Mr. R—— shew that he is a sincere enquirer after scientific knowledge, by frankly admitting the truth of these propositions; and then I promise him that I will rigorously analyze every sentence of his paper, and demonstrate the absurdity of his assertions, reasonings, and conclusions. He declines to admit that π cannot be less than $3\frac{1}{8}$, nor greater than $3\frac{1}{5}$. Why should he? If Mr. R—— decline to admit these propositions, with my experience of that gentleman, it would be folly to continue the controversy. In doing so I should simply prove myself to be no more fit to deal with such an opponent, than an unskilful fisherman to handle a powerful and slippery eel.

I can assure you, my dear Sir, it is painful for me to write thus plainly, but truth admits of no compromise, and the course pursued by Mr. R—— forces it upon me.

Mrs. Smith joins me in kind regards, and we are glad to hear Mrs. S—— is so much better as to be able to sit up for eight hours.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. R——'s PAPER, *7th March*, 1868.

Mr. Smith has transmitted a list of eight "propositions," saying that it "would be an act of folly on his part to attempt to analyze" my last paper, "unless, and until he knows," that I agree to them.

If Mr. Smith finds, in any of my notes a single word that seems to imply my disagreement with any one of these eight propositions, it must have been put down by inadvertency. These propositions are the merest common places, and may be reduced to three or four; (First) $\pi\ r^2\ =$ area of circle. (Second) $\frac{\pi}{3}\ \times$ (perimeter of a regular hexagon) = circumference. Perimeter of hexagon being $6\ r$ or $3\ d$, this equation becomes $\frac{\pi}{3} \times 3\ d = \pi\ d =$ circle. So (Third) $\frac{\pi}{3} \times$ (area of a regular dodecagon) $= \frac{\pi}{3} \times 3\ r^2 = \pi\ r^2 =$ area: Fifth, sixth, seventh, and eighth, are merely particular applications of the general formula. As to the fourth, I am puzzled to see what it has to do with the matter; but if I understand it in the same sense as Mr. Smith, how could a sane man but say yes to it? The angles in similar right angled triangles are the *same angles*, and, of course, the natural sine of an angle of 25° in one triangle is $=$ the natural sine of an angle of 25° in any and every other triangle.

It is for Mr. Smith to show why he defers or refuses to establish his premisses until he know whether I agree to these simple matters; or, why he makes it a matter of doubt, whether I shall decline to admit them. Why did he not tell me sooner that I seemed to doubt or ignore them? Hitherto I have withheld assent to Mr. Smith's reasonings, simply because the premisses are not self-evident, and need proof. This proof has been withheld hitherto, and I must confess to a curiosity to see it. *

* I quote the following from my Letter of the 25th January, to Mr. S——:

"Permit me to refer you to the Foot-note on page 57 of my Letter to the Duke of Buccleuch. Now, I will show you how to construct the diagram, and then make use of it to point out a remarkable Trigonometrical paradox.

"Well then, I construct the enclosed diagram (*see Diagram* III.) in the following way. I draw two straight lines at right angles, making B the right angle. I then open my compasses, and from the angle B mark off three equal parts, making them together equal to A B, and from B I mark off four of such equal parts, together equal to B C, and join A C. With B as centre, and B C as interval, I describe the circle X. With C as centre, and C B as interval, I describe the circle Y. With C as centre, and C A as interval, I describe the circle Z. With B as centre, and the same interval, I describe the circle X Y. I produce B A to meet the circumference of the circle X Y at the point D, and join D C and O C. With C as centre, and C D as interval, I describe the circle X Z. I produce C A to meet the circumference of the circle X Z at the point E, and let fall a perpendicular, to meet the line C B, produced at F. From the point *n* where the circumference of the circle Z cuts the line F C, I draw *n m* tangental to the circle Z, to meet the line E C at the point *m*.

Let $A B = 3$, and $B C = 4$. Then: $A C = 5$, $D B = 5$, and $D C = \sqrt{41}$. Hence: D B C is a similar triangle to the triangle E O G in Mr. R——'s figure. Now, $E C = D C$, for they are radii of the same circle. But, E F is obviously a shorter line than O B, and $O B = B C$, for they are radii of the same circle, and F and B are right angles. Hence: E F C and D B C are right angled triangles, and have their sides that subtend the right angle equal, but they are not similar and equal triangles. Now, $\frac{A B}{A C} = \frac{3}{5} = \cdot 6$ is the

Mr. JAMES SMITH *to* Mr. Jno. S——.

BARKELEY HOUSE, SEAFORTH,
9th March, 1868.

MY DEAR SIR,

About a week ago, very much to my astonishment, I received a letter from my old correspondent, the Rev. Geo. B. Gibbons, of which the following is a copy:—

LANEAST, LAUNCESTON,
27th February, 1868.

DEAR SIR,

I have seen advertised several times, in the *Daily Telegraph Newspaper*:—

natural sine of the angle C, in the triangle A B C; and $\frac{BC}{AC} = \frac{4}{5} =$ $\cdot 8$ is the natural sine of the angle A. But, $Cn = CA = 5$, for they are radii of the same circle, and $m\,n$ C and A B C are similar right angled triangles; therefore, $\frac{3}{4}(nC) = \frac{3}{4}(5) = 3{\cdot}75 = mn$, and $\frac{5}{4}(nC) = \frac{5}{4}(5)$ $= 6{\cdot}25 = mC$; therefore, $\frac{mn}{mC} = \frac{3{\cdot}75}{6{\cdot}25} = {\cdot}6$, and, $\frac{nC}{mC} = \frac{5}{6{\cdot}25} = {\cdot}8$, and the sine and co-sine of the angles m and C in the triangle $m\,n$ C, are the same as the sine and co-sine of the angles A and C, in the triangle A B C.

"Now, E C $= \sqrt{41} = 6{\cdot}403124$. Well, then, by hypothesis, let the obtuse angle E, in the right angled triangle E F C, be an angle of $53°\ 8'$, and the side E C, subtending the right angle, $6{\cdot}403124$ miles in length, and be given to find the lengths of the sides E F and F C, which contain the right angle. Then: $90° - 53°\ 8' =$ $36°\ 52'$ = the angle C.

Then: by Hutton's Tables.

As Sin. of angle F = Sin. 90°... ...	... Log.	10·0000000
: the given side E C = 6·403124 miles...	... Log.	0·8063919
: : Sin. of angle C = Sin. 36° 32′ ...	... Log.	9·7781186
		10·5845105
		10·0000000
: the required side E F = 3·841586 miles	... Log.	0·5845105

"The Circle Squared." By Edward Thornton.
London: Edward Stanford, 6, Charing Cross.

No price is mentioned, and therefore I have not ordered the book. Have you seen it? And does Mr. Thornton's determination agree with yours?

Believe me,
Yours very truly,
G. B. GIBBONS.

"Again:

As Sin. of angle F = Sin. 90°... ...	... Log.	10·0000000
: the given side E C = 6·403124 miles...	... Log.	0·8063919
: : Sin. of angle E = Sin. 53° 8′ ...	... Log.	9·9031084
		10·7095003
		10·0000000
: the required side F C = 5·122716 miles	... Log.	0·7095003

"Then: $\frac{EF}{EC} = \frac{3·841586}{6·403124}$ = ·5999549 is the natural sine of the angle C, and the natural co-sine of the angle E. $\frac{FC}{EC} = \frac{5·122716}{6·403124}$ = ·8000338 is the natural sine of the angle E, and natural co-sine of the angle C. Now, ·5999549 and ·8000338 are the natural sines and co-sines of angles of 36° 52′ and 53° 8′, as given in Hutton's Tables.

"Well, then, the triangles ABC, *mn*C, and EFC are similar right angled triangles, and I have proved that the natural sines and natural co-sines in the two former are ·6 and ·8. But, according to Hutton (and his Tables are not at variance with others), the natural sine of the angle C, in the triangle E F C, is less than ·6, and the natural co-sine greater than ·8. Can the natural sines and co-sines be different in similar right angled triangles? Can Trigonometry and Geometry be inconsistent with each other? Is not this a paradox? Will Mr. R—— be good enough to unravel it? By making a few additions to this geometrical figure, a pamphlet would not be sufficient to exhaust its properties.

"In conclusion, I shall be curious to know how Mr. R—— will next proceed to convince me that he has demonstrated in my own way, that the true value of π is $3\frac{7}{25}$: and still more curious to see how he will unravel the paradox,

I acknowledged the receipt of this communication, informing Mr. Gibbons that I had not heard of Mr. Thornton's work, but that I should order it the first time I was in Liverpool; and I took the opportunity of giving him the same proofs of the value of π, by means of continued proportion, that I gave to your brother in my Letter to him of the 22nd February.

I got Mr. Thornton's work, and on Saturday sent it to Mr. Gibbons, with a Letter of which the following is a copy.

BARKELEY HOUSE, SEAFORTH,
7th March, 1868.

DEAR SIR,

I send you by this day's post a copy of Mr. Edward Thornton's work on the Quadrature of the Circle. On perusal, you will find that his square Y is not equal in area to the circles A, but exactly equal to a regular dodecagon inscribed within them. He seems to have lost sight of the fact, that if we inscribe 6, 12, 24, 48 sided regular polygons—and so on as far as we please—within a circle, the diameter of all these polygons will be exactly equal to the diameter of the circle.

I may take this opportunity of shewing you a very pretty geometrical method of squaring the circle, and demonstrating the quadrature by some most interesting mathematical proofs, all of which are in perfect harmony with the proofs I gave you of the value of π by continued proportion, in my Letter of the 29th ult.

In the enclosed diagram, *(See Diagram I.)* if the line $CD = 4$, then $CE = 5$, $CR = 7$, DE and $RL = 1$, and $CL = 8$, by construction. But, $(CD + DL)$ is obviously equal to $(PI + IO)$, that is $= OP$: and OP is the

diameter of the circle Y. Hence: The diameter of the circle Y = twice the diameter of the circle X = 2 (C D).

Well then, join T R and V W; that is, draw the diagonals of the rectangles TL·LR and VN·NW; and on the diagonals TR and VW describe squares. These squares will stand on the circle Y, and are exactly equal to the circle in superficial area. Hence: $TR^2 = VW^2$, and this equation $= 3\frac{1}{8} (DC^2)$.

Now, (P I + I O) is a binomial, of which the two terms P I and I O are in the ratio of 7 to 1, by construction, and the sum of the two terms = O P, the diameter of the circle Y. Hence, we get the following equations.

First: $(PI^2 + IO^2) = 3\frac{1}{8} (DC^2)$, and this equation = area of the circle Y.

Second: $(PI^2 - IO^2) = 6 \left(DC \times \frac{DC}{2}\right)$, and this equation = area of an inscribed regular dodecagon to the circle Y.

Third: D B is a diagonal of the square A B C D. Hence: $(DC^2 + CB^2 + DB^2) = 4 (DC^2)$, and this equation = area of the circumscribing square to the circle Y.

Hence: the area of the square on E C, that is, the area of the square E C G F = half the area of the circle Y, and twice the area of the circle X; and it follows of necessity that $\frac{1}{8}$th part of the circumference of any circle = side of square containing half the area of a circle; and $\frac{1}{8}$th part of the perimeter of any square = diameter of a circle containing half the area of the square.

Now, let E C a side of the square E C G F be represented by the square root of an arithmetical quantity, say $\sqrt{7}$; and be given to find the diameter of the circle Y.

Then: $EC^2 \times \frac{8}{\pi} = EC^2 \times \frac{8}{3{\cdot}125} = 7 \times 2{\cdot}56 = 17{\cdot}92$ and $\sqrt{17{\cdot}92}$ is the diameter of the circle Y. Again: Let $EC = \sqrt{7}$, and be given to find the diameter of the circle X. Then: $\frac{EC^2}{\frac{1}{2}(\pi)} = \frac{7}{1{\cdot}5625} = 4{\cdot}48$; and $\sqrt{4{\cdot}48}$ is the diameter of the circle X. Proof: $\frac{1}{2}(\sqrt{4{\cdot}48}) = \sqrt{1{\cdot}12}$ = radius of the circle X; and $\pi\,(r^2)$ = area in every circle; therefore, $\pi\,(\sqrt{1{\cdot}12^2}) = 3{\cdot}125 \times 1{\cdot}12 = 3{\cdot}5$ = area of the circle, and is equal to half the area of the square E C G F. Again $\frac{1}{2}(\sqrt{17{\cdot}92}) = \sqrt{4{\cdot}48}$ = radius of the circle Y; therefore, $\pi\,(r^2) = 3{\cdot}125 \times 4{\cdot}48 = 14$ = area of the circle Y, and is equal to twice the area of the square E C G F; and 4 times the area of the eircle X. Hence: $\sqrt{EC^2 \times \frac{8}{\pi}} = \frac{8}{5}(EC)$, and this equation = $\sqrt{17{\cdot}92}$ = diameter of the circle Y, when $EC = \sqrt{7}$; and $\sqrt{\left(\frac{EC^2}{\frac{1}{2}(\pi)}\right)} = \frac{4}{5}(EC)$, and this equation = $\sqrt{4{\cdot}48}$ = diameter of the circle X, when E C a side of the square E C G F = $\sqrt{7}$.

Now, $\frac{\pi}{4} = 12{\cdot}5\left(\frac{\pi}{50}\right)$, that is, $\frac{3{\cdot}125}{4} = 12{\cdot}5 \times {\cdot}0625$, and this equation = area of the circle Y, when O P the diameter of the circle Y = 1. That is, $3\frac{1}{8}\left(\frac{1}{2}\right)^2 = 3{\cdot}125 \times {\cdot}25$ = ${\cdot}78125$ = area of the circle. But the equation, $\frac{\pi}{4} = 12{\cdot}5\left(\frac{\pi}{50}\right)$ = area of a circle of diameter unity = 1, whatever be the value of π, and this fact may be proved by means of any *hypothetical* value of π. I now put the question to you, as I have put it to myself: What necessarily follows from these facts? I answer: It necessarily follows that, to

prove the orthodox value of π to be true, we must connect it with some proof by continued proportion, similar to that to which I directed your attention in my last Letter ; and we must also make it harmonize with such-like truths as those to which I have now directed your attention, in connection with the enclosed diagram. Can we connect $\pi = 3{\cdot}14159\ldots$ with anything, or make it harmonize with anything? In how many ways have I connected $\pi = 3{\cdot}125$, with practical geometry?

With these facts before you, I venture to say, not intending it either boastingly or offensively, that you may as well attempt to prove the square of 7 greater than 49, as attempt to controvert the ***theory*** that 8 circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3{\cdot}125$, the true arithmetical value of π.

Believe me, my dear Sir,

Very truly yours,

The Rev. Geo. B. Gibbons, B.A., JAMES SMITH.

Laneast.

These truths are far from exhausting the properties of the enclosed geometrical figure, and I will give *you* one proof of it.

Now, KE = PC = DC = DL, by construction ; and, CL — CE = DL — DE = EL, by construction. But, CE = $\frac{5}{4}$ (DC), by construction ; therefore, EL = $\frac{3}{4}$ (DC), by construction. Hence : KE = radius of the circle Y; and I have proved that the radius of the circle Y = $\sqrt{4{\cdot}48}$ when EC a side of the square ECGF = $\sqrt{7}$; therefore, $\frac{3}{4}$ (KE) = $\frac{3}{4}$ ($\sqrt{4{\cdot}48}$) = $\sqrt{2{\cdot}52}$ = EL; therefore, ($KE^2 + EL^2 + KL^2$) = $4{\cdot}48 + 2{\cdot}52 + 7 = 14 = \pi(r^2)$;

that is, $= 3{\cdot}125 \times 4{\cdot}48 = 14 =$ area of the circle Y. Hence: The sum of the areas of squares on the sides of the rectangles OKEL or OKHM $= \{2(KE^2) + 2(EL^2)\}$ or, $\{2(KH^2) + 2(HM^2)\} = (8{\cdot}96 + 5{\cdot}04) = 14 =$ area of the circle Y, when E C a side of the square ECGF $=$ $\sqrt{7}$.

Now, my dear Sir, is it possible that any right reasoning man, moderately versed in Geometry and Mathematics, can—in the face of these facts—have an *honest* doubt on his mind, as to the true value of π; or the true ratio of diameter to circumference in a circle?

I send you herewith a copy of Mr. E. Thornton's book on circle-squaring. Saturday's post brought me a pamphlet on this subject from the pen of a Mr. Lawrence L. Benson, of South Carolina, United States of America, who was so confident that he had squared the circle, that he came over to this country purposely to enlighten our geometrical and mathematical *savans;* and I may tell you that he has fallen into precisely the same blunder as Mr. Edward Thornton.*

Believe me, my dear Sir,
Very truly yours
JAMES SMITH.

Jno. S——, Esq.

* Mr. Gibbons returned me the Thornton Pamphlet, with a short note of thanks, but did not make the slightest reference to the demonstrations given in my two Letters.

Mr. SMITH *to* Mr. Jno. S——.

BARKELEY HOUSE, SEAFORTH,
10*th March*, 1868.

MY DEAR SIR,

By some very simple additions to the diagram enclosed in my Letter of yesterday, we may obtain a demonstration, so plain and palpable, that the true ratio of diameter to circumference in a circle is "*un fait accompli*," as will set at defiance all the *chicanery* of the Mathematician to disturb it: not only so, but even without these additions to the diagram, we may furnish a proof, that will put the question of the solution of the problem of *Squaring the Circle* beyond the pale of further controversy. It might be said, I was not dealing fairly with opponents, if I hesitated to furnish my proofs, and so I shall proceed to direct your attention to them. Before doing so, however, I beg to make a few preliminary observations.

Well, then, referring you to the diagram enclosed in my yesterday's Letter *(see Diagram I.)*, let a denote the area of the circle X. Let b denote the area of an inscribed circle to the square ECGF. And, let c denote the area of the circle Y. Then: Because the areas of circles are to each other as the squares of their diameters; or, in other words, since the areas of circles are to each other as the areas of their circumscribing squares; it follows of necessity, that, $CD^2 : a :: CE^2 : b$; and, $CE^2 : b :: CL^2 : c$. Again: Because, $CD : CE$ in the ratio of 4 to 5, and, $CE : CL$ in the ratio of 5 to 8, by construction; it follows of necessity—and π is not an element in the proof—that, $16 : 25 :: a : b$; and, $25 : 64 :: b : c$.

For example: Let a be represented by any arithmetical quantity, say 15. Then : 16 : 25 : : 15 : 23·4375 ; therefore, b = 23·4375 ; and, 25 : 64 : : 23·4375 : 60; therefore, c = 60 = 4 times a. Hence: It follows of necessity, that the true value of π *must*, and that none but the true value of π *will*, harmonize with these facts: not only so, but it also follows, that π cannot be otherwise than a *determinate* arithmetical quantity.

Now, $\{(b - \frac{1}{5}b) - \frac{1}{5}(b - \frac{1}{5}b)\} = [\{23{\cdot}4375 - \frac{1}{5}(23{\cdot}4375)\} - \frac{1}{5}(18{\cdot}75)\}] = (18{\cdot}75 - 3{\cdot}75) = 15 = a$; that is, = the given area of the circle X. Hence: If we inscribe a circle in the square E C G F, and in this circle inscribe a square, this square and the circle X are exactly equal in superficial area. Proof: The area of a circumscribing square to any circle is equal to twice the area of an inscribed square; therefore, 2 (15) = 30 = area of the square E C G F ; therefore, $\sqrt{30}$ = diameter of an inscribed circle ; that is, = diameter of a circle of which b (= 23·4375,) is the area ; therefore, $\frac{1}{2}(\sqrt{30},) = \sqrt{7{\cdot}5}$ = radius of the circle ; and $\pi\ (r^2)$ = area in every circle. But, $3{\cdot}125\ (\sqrt{7{\cdot}5^2}) = 3{\cdot}125 \times 7{\cdot}5 = 23{\cdot}4375 = b$; and since $\sqrt{7{\cdot}5^2}$ can be multiplied by no other arithmetical symbols but 3·125 and produce this result, it follows of necessity, that 3·125 must be the true arithmetical value of π, and makes 8 circumferences = 25 diameters in every circle. I think you cannot fail to perceive how this proof of the value of π, harmonizes with that I have given you in the examples by continued proportion.

The enclosed diagram (*See Diagram* IV.) is a fac-simile of that contained in my Letter of yesterday, with the following additions. Join O D, and on O D describe the square O D P S, and in this square inscribe the circle Z.

Produce O D to B, bisecting the square A B C D and its inscribed circle X.

Now, because the circle Z is an inscribed circle to the square O D P S, the area of the circle Z = half the area of the circle Y ; and twice the area of the circle X. Well then, let the area of the square O D P S be represented by any arithmetical quantity, say 60. Then: $\{(60 + \frac{1}{4} 60) + \frac{1}{4}(60 + \frac{1}{4} 60)\} = (75 + 18{\cdot}75) = 93{\cdot}75$ = area of the circle Y ; therefore, $\frac{93{\cdot}75}{2} = 46{\cdot}875$ = area of the circle Z. Then: $\frac{93{\cdot}75}{\frac{1}{4}\pi} = 93{\cdot}75 \times 1{\cdot}28 = 120$, and this equation = area of the square L C N M circumscribed about the circle Y ; therefore, O P the diameter of the circle Y = $\sqrt{120}$. But, L C a side of the circumscribing square to the circle Y = O P, and L C : E C a side of the square E C G F, in the ratio of 8 to 5, by construction ; therefore, $\frac{5}{8}(L\,C) = \frac{5}{8}(\sqrt{120}) = \sqrt{(\frac{5^2}{8^2} \times 120)} = \sqrt{(\frac{25}{64} \times 120)} = \sqrt{(\cdot 390625 \times 120)} = \sqrt{46{\cdot}875}$ = E C ; therefore, E C² = $\sqrt{46{\cdot}875}^2 = 46{\cdot}875$ = area of the square E C G F, and is exactly equal to the area of the circle Z. This again beautifully harmonizes with my proofs of the value of π by continued proportion ; and with practical geometry in a thousand ways.

Well, then, this proof puts the question of the solution of the problem of *Squaring the Circle* beyond the pale of controversy, and it would be worse than folly to continue a discussion with any Mathematician who would attempt to dispute it.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

Jno. S——, Esq.

MR. SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH.
12th March, 1868.

MY DEAR SIR,

Your favour of the 10th instant is to hand, enclosing Mr. R——'s Paper in reply to my Letter of the 4th instant, and I thank you very sincerely for the promptitude with which you perform your part, as medium between that gentleman and your humble servant.

Since my last to you, I have written two Letters to your brother, which I presume Mr. R—— will have the opportunity of perusing: if so, *he* will find that I have *established my premisses*, and that he cannot even dispute them, without falsifying one or other of the eight propositions, which he distinctly admits to be *axiomatic*, in his present Paper. Hence the importance of having these propositions admitted! He observes:—"*As to the fourth, I am puzzled to see what it has to do with the matter.*" In explanation, I beg to call Mr. R——'s attention to the following quotation from my Letter to you of the 25th January. "Now, the triangles ABC, *mn*C, and EFC are similar right-angled triangles *(see Diagram III)*, and I have proved that the natural sines and natural co-sines in the two former are ·6 and ·8. But, according to Hutton (and his Tables are not at variance with others) the natural sine of the angle C, in the triangle E F C, is less than ·6, and the natural co-sine—greater than ·8. Can the natural sines and co-sines be different in similar right angled triangles? Can Trigonometry and Geometry be inconsistent with each other? Is not this a paradox? Will Mr. R—— be good enough to unravel it?" Mr. R——

has never referred to this quotation in any way, but he cannot fail to perceive, that the question at issue between us is deeply involved in it, unless he is prepared to *prove* that, $\frac{\pi}{300}$ does not express the circular measure of an angle of 36 minutes. *Had* Mr. R—— read my Letter to the Duke of Buccleuch with *care*, he would have seen, that my *object* was to *shew*, that the arithmetical value of the *natural*, *geometrical*, and *trigonometrical* sine of an angle—*which mathematicians maintain to be the same thing in all angles*—may be all different; quite as much as to *shew*, that a square equal to a given circle, "*may, and does exist*," and can be isolated and exhibited.*

In his previous Paper, Mr. R—— observes:— "*He* (Mr. Smith) *says that I am a firm believer in the orthodox* π. *I may ask when I have said that? I have said that all* Mr Smith's *attempts to convince me that* $\pi = 3\frac{1}{8}$, *serve to rather convince me that* π *is not determinate. I further avow my belief in the correctness of the methods by which Mathematicians have sought the value of it.*" Now, I have never disputed the calculations by which a Mathematician makes $\pi = 3{\cdot}14159265\ldots$ What I say is this. These calculations are based on an *assumption*, which *assumption* never

* At the time of writing my Letter of the 25th January, to Mr. S——, I might have put the following question to Mr. R——: Can there be two dissimilar and unequal right-angled triangles, having the sides that subtend the right angle equal? If he had answered this question at all, I have little hesitation in saying his answer would have been—*Certainly not! If, in a right-angled triangle, the sides subtending the right-angle are equal, the other sides and angles must necessarily be equal.* In the Foot-note on page 43, I have demonstrated that this is not a fact; and my demonstration exposes the absurdity of Mathematicians rambling round a curved line, by polygons, in their search after π.

has been, and never can be proved. Well, then, since Mr. R—— believes in the correctness of the Mathematician's methods of searching for π, and as I admit the correctness of the figures deduced by these methods, there can be no dispute between me and Mr. R—— as to the fact, that π cannot be less than $3\frac{1}{8}$, nor so great as $3\frac{1}{7}$. Hence, any argument based on a value of π outside these limits is inadmissible under the following circumstances. It has been proved by Mr. R——, that, diameter and area of a circle are represented by the same arithmetical symbols, when circumference of the circle $= 4$; and, $3\frac{1}{8} \times 1\frac{7}{25}$, that is, $3{\cdot}125 \times 1{\cdot}28 = 4$. Now, Mr. R—— catches at, and plays upon, the following quotation from one of my Letters. "*Now since no other arithmetical symbols than* $3\frac{1}{8} \times 1\frac{7}{25}$ *will produce* 4, *it follows of necessity, that* $3\frac{1}{8}$ *must be the true value of* π." Taking what precedes it in connection with this quotation, it is difficult to conceive that Mr. R—— could mistake my meaning. I admit, however, that in this quotation I have not well expressed what I meant. I should have said :—Now, since no other arithmetical symbols, (*within admissible limits*) will multiply into $1{\cdot}28$ and produce 4, it follows of necessity, that $3{\cdot}125$ must be the true value of π. Had I so expressed my meaning, I think I should have spared Mr. R—— and myself *some* trouble.

Mr. R—— says that my process "*proceeds on the principle that* π *is determinate.*" Granted ! I have given a proof that π must be determinate, in my last Letter to your brother. He then observes:— "*Putting his* (Mr. Smith's) *reasoning as a syllogism, it stands thus: The ratio between the circumference and the diameter is a determinate quantity.* $3\frac{1}{8}$ *is the only determinate quantity that will give a 'finite' quotient with* (4). *Therefore* $3\frac{1}{8}$ *is*

the ratio in question." Now, if the second proposition had run thus:—$3\frac{1}{8}$ is the only determinate quantity—*within admissible limits*—that will give a finite quotient with (4); I should like to know how Mr. R—— would have gone to work to controvert the conclusion.

In another part of his Paper, Mr. R—— observes:—"*He* (Mr. Smith) *says that only one of my* 5 *numbers can be expressed by a terminable decimal, and assumes that this puts the others out of court. This is to introduce a new and arbitrary condition, and the necessity of this is not established by* Mr. Smith. *Are not* $3\frac{4}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, $3\frac{1}{3}$ *all as determinate as* $3\frac{1}{6}$ *and* $3\frac{1}{8}$! *How can this condition be justified? Mathematics have always been called the exactest of sciences. But this way of introducing new conditions by mere assertion, makes exact treatment impossible. If the condition was necessary,* Mr. Smith *should have stated it in his process. The condition necessitates another syllogism, which takes the following shape:—*

"*The ratio between circumference and diameter must be expressed in a determinate decimal.* $3\frac{1}{8}$ *is the only number that obeys this condition; therefore,* $3\frac{1}{8} = \pi$. *Now, let* Mr. Smith *justify the major: I dispute it, and ask proof. But, moreover, by* Mr. Smith's *own confession* $3\frac{1}{6}$ *also meets the condition he lays down, and therefore* $3\frac{1}{8}$ *is not the only number, and the minor is not true. Is this reasoning?*"

This shews that I should have been careful to extract from Mr. R—— at the outset, the admission that π cannot be less than 3·124 or greater than 3·199, and consequently, that the value of π must necessarily be intermediate between these two quantities.

Now, if the second proposition of the latter syllogism had ran thus:—$3\frac{1}{8}$ is the only number—*within admissible*

limits—that obeys this condition. Would not all Mr. R——'s 5 numbers have been put "*out of court?*" and how would he have gone to work to controvert the conclusion?

I could say much more to Mr. R—— with reference to his Paper, but I shall wait and see whether my last Letter to your brother, convinces him and Mr. R—— that π is a determinate quantity.

I am glad to hear Mrs S—— goes on improving, and with kind regards to you and all your family circle,

Believe me, my dear Sir,

Very faithfully yours,

JAMES SMITH.

J—— S——, Esq.

BARKELEY HOUSE, SEAFORTH,
19*th March*, 1868.
(Posted 24th, 1868.)

MY DEAR SIR,

I was in Liverpool to-day, and a copy of the "*Hull and Eastern Counties Herald*," of the 27th February, was presented to me by a friend, himself an Astronomer and Mathematician, in which there appears a Letter from the pen of the Hull circle-squarer, of which the following is a copy. The paper was sent to my friend, by a brother of Mr. Harbord, for the purpose of being handed over to me after perusal, not knowing that my friend has privately admitted to me, that I have solved the knotty question of the true ratio of diameter to circumference in a circle, beyond the possibility of dipute by any *honest* Mathematician.

"THE CIRCLE SQUARED."

To the Editor of the Herald,

SIR,

I am induced to solicit space in your valuable Journal, having to submit for consideration the following demonstration of the "Circle Squared" in as concise a form as possible, trusting that it will satisfy those who need information on the subject, and be approved by others learned in Mathematics. As it is not my intention to dilate on the question of "accepted ratios" and "approximate tables" in use at the present time, I give, as understated, the result of my own calculations, all of which are final and conclusive.

Circumference of circle (diameter one). Root of Circumference.

3·141592653589381932399749I6.—1·7724538509054

```
  1

27 / 214
      189
347 / 2515
       2429
3542 / 8692
        7084
35444 / 160865
         141776
354485 / 1908935
          1772425
3544903 / 13651089
           10634709
35449068 / 301638038
            283592544
354490765 / 1804549419
             1772453825
35449077009 / 320955943239
               319041693081
3544907701805 / 19142501589749
                 17724538509025
35449077018104 / 1417963080724I6
                  1417963080724I6
```

Root of Circumference:

```
                    1·7724538509054
                                 ·5 Radius
                    ──────────────────
                    ·8862269254527 Side of Square.
                    8862269254527
                    ──────────────
                    62035884781689
                   17724538509054
                  44311346272635
                 35449077018108
                44311346272635
               17724538509054
              79760423290743
             53173615527162
            17724538509054
           17724538509054
          53173615527162
         70898154036216
        70898154036216
        ──────────────────────────────
        ·785398163397345483099993729 Area and Quad. of Circle
                                   4
        ──────────────────────────────
        3·141592653589381932399749 16  Circum. of Circle.
                                1·25   Ratio
        ──────────────────────────────
      1570796326794690966199874580
      628318530717876386479949832
      314159265358938193239974916
      ──────────────────────────────
      3·926990816986727415499686450o Ratio.
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I am, Sir, Yours respectfully,

H. HARBORD.

Hull, 24th February, 1868.

You will observe that Mr. Harbord's fancied solution of the "*Circle Squared*" is based on the *assumption* that π, that is, the circumference of a circle of diameter *one*, must be a *determinate* quantity. This is one of the charges Mr. R—— brings against me. Mr. Harbord gives us

a string of 26 decimals following the integer 3, the square root of which quantity is finite. How he gets these figures he does not condescend to tell us; but, having found them, he jumps to the conclusion that he has discovered the true ratio of diameter to circumference in a circle. What he means by the word *ratio*, one has some difficulty in defining, from the way he employs it; but, it appears to me, that the following must be his meaning.

Final ratio : ratio 1·25 : : π : 1; that is,

3·92699081698672741549968645 : 1·25 : :

3·14159265358938193239974916 : 1.

If I am right in my conjecture, there can be nothing in his solution; for, we may concoct values of π by the hundred that will produce a similar result. Take the following example as a proof:—

Circum. of Circle (diam. 1.)	Root of Circum.	
3·125824 ———	1·768	
1	·5	Radius.
27 / 212	·8840	Side of Square.
189	·8840	
346 / 2358	353600	
2076	70720	
3528 / 28224	70720	
28224	·78145600	Area and Quad. of Circle.
	4	
	3·12582400	Circum. of Circle.
	1·25	
	1562912000	
	625164800	
	312582400	
	3·9072800000	Ratio.

Hence: 3·90728 : 1·25 : : 3·125824 : 1; therefore, $\frac{3·90728}{1·25}$ and $\frac{3·125824}{1}$ are equivalent ratios, and both would express the ratio of circumference to diameter in a circle, if 3·125824 were the true circumference of a circle of diameter = 1.

Now, I do not dispute the correctness of Mr. Harbord's calculations, nor can he dispute mine, and the simplicity of my example, as compared with his, will be obvious to any third class school-boy. Well, then, $12·5\left(\frac{3·125824}{50}\right)$

$$= \frac{12·5 \times 3·125824}{50} = \frac{39·0728}{50} = \frac{3·125824}{4} = ·781456 =$$

area of a circle of diameter *one*, on the *hypothesis*, that π = 3·125824. The importance of this fact is not self-evident, and I venture to give you another example, before directing your attention to the results that follow:—

Circum. of circle (diameter one).	Root of circum.	
3·0976 ————	1·76	
1	·5	Radius.
27 / 209	·880	Side of Square.
189	·880	
346 / 2076	70400	
2076	7040	
	·774400	Area and Quadrature of circle.
	4	
	3·097600	Circumference of Circle.
	1·25	
	15488000	
	6195200	
	3097600	
	3·87200000	Ratio.

Hence: $3·872 : 1·25 :: 3·0976 : 1$; therefore, $\frac{3·872}{1·25}$ and $\frac{3·0976}{1}$ are equivalent ratios, and both would express the ratio of circumference to diameter in a circle, if 3·0976 were the true value of the circumference of a circle of diameter = 1. Well, then, $12·5 \left(\frac{3·0976}{50}\right) = \frac{12·5 \times 3·0976}{50}$ $= \frac{38.72}{50} = \frac{3·0976}{4} = ·7744$ = area of a circle of diameter *one*, on the *hypothesis*, that $\pi = 3·0976$; and, as I have already stated, we may find values of π by the hundred, that will produce similar results.

I fancy I hear Mr. R—— exclaim: *Cui bono!* No doubt both you and Mr. Harbord have *hit* upon values of π, by which the calculations work out to final results; but, what does either Mr. Harbord's elaborate calculations or your more simple ones prove? What do we learn from them? Do they teach us anything bearing upon the question at issue; that is to say, bearing on the value of π, or the true ratio of diameter to circumference in a circle? I answer, yes! These facts teach us more than was ever dreamed of in the philosophy of our recognised mathematical authorities! Well, then, they teach us:—

First: In a circle of diameter $= 1$, $r^2 (\sqrt{\pi^2}) = \frac{\pi}{4}$. But, $\frac{\pi}{4} = 12·5 \left(\frac{\pi}{50}\right)$, whatever be the value of π. Hence: If the diameter of a circle $= 1$, then, $12·5 \left(\frac{\pi}{50}\right) = r^2(\sqrt{\pi^2})$*

* $r^2 (\sqrt{\pi^2})$ and $\pi (r^2)$ are equivalent expressions. But, if we first extract the square root of π, to get the arithmetical value of the equation $12·5 \left(\frac{\pi}{50}\right) = r^2 (\sqrt{\pi^2})$, we should make one side of the equation commensurable and the other side incommensurable, whether we adopt 3·1416, 3·14159265, or 3·125 as the value of π.

and this equation = area of the circle, whatever be the value of π. From these facts, it follows of nécessity, that if c denote the circumference, and a denote the area of a circle, $c + a = \frac{10\,(a)}{2}$, when the diameter of the circle = 1, whatever be the value of π.

Second: Mr. Harbord, in the original announcement of his vaunted quadrature, expressed what he termed his "*ratios,*" by the fraction $\frac{3{\cdot}92699081698672741549968645}{1{\cdot}25}$. The numerator of this fraction is the same as the final figures—called ratio—in his elaborate calculations; and—for want of a better term, and indeed, in a Letter I have seen of his, he so calls it—we may denominate the numerator of this fraction the "*Final ratio.*" Now, let $\sqrt{2\,(1{\cdot}25)}$, that is, the square root of twice the denominator of this fraction $= \sqrt{2{\cdot}5}$, represent the side of a square. Then: $\sqrt{2{\cdot}5}^2 \times \frac{\pi}{4}$ = area of an inscribed circle, and is exactly equal to $\frac{\text{Final ratio}}{2}$, whatever be the value of π. For example: Let $\sqrt{2{\cdot}5}$ represent the side of a square; and, *by hypothesis*, let $\pi = 3{\cdot}125824$. Then: $\sqrt{2{\cdot}5}^2 \times \frac{\pi}{4} = 2{\cdot}5 \times {\cdot}781456 = 1{\cdot}95364$; that is, $= \frac{3{\cdot}90728}{2} = \frac{\text{Final ratio}}{2}$; as you cannot fail to perceive by referring to sheet 5 (*see page* 61) of this communication. Hence: If we work out the calculations, on $\pi = 3{\cdot}0976$, or, by Mr. Harbord's 26 decimal π, we obtain the same result; that is, $\frac{\text{Final ratio}}{2}$; therefore, $\pi = 3{\cdot}125824$, or, $\pi = 3{\cdot}0976$, are just as good, and far more simple, than Mr. Harbord's six-and-twenty

decimal π, for the purpose of demonstrating, that $\sqrt{2 \cdot 5^2} \times \frac{\pi}{4} = \frac{\text{Final ratio.}}{2}$

Now, if we make the "*First*" the major, and the "*Second*" the minor premiss of a sollogism; the third proposition or conclusion follows of necessity, viz.: whatever be its value, π ***must be*** a determinate arithmetical quantity.

Now, my dear Sir, the third proposition, or conclusion, is incontrovertible. But, it is one of those things that Mathematicians and Astronomers do not ***wish*** to see; which it does not ***suit*** them to see; and, therefore, which they have ***resolved*** not to see. And I now suspect I shall find your relative, Mr. R——, no exception to the rule. They know—perhaps too well—that to admit the incontrovertibility of the third proposition of my argument, stated syllogistically, would subvert existing systems, both in Mathematics and Astronomy. At the first meeting, held in Liverpool, of "*The British Association for the Advancement of Science*," a gentleman personally known to me, and still living, proposed to read a Paper in the Physical Section, which threw doubts on existing Astronomical theories. He was politely told that he could not be permitted to read such a Paper, and the late Dr. Whewell gave as the reason:—"*We don't wish to disturb existing systems.*" More than 30 years, have not made Mathematicians and Astronomers either wiser or better men.

Now, $12 \cdot 5 \left(\frac{\pi}{50}\right) = r^2(\sqrt{\pi^2})$, in a circle of diameter $= 1$; and $\sqrt{2 \cdot 5^2} \times \frac{\pi}{4} = \frac{\text{Final ratio}}{2}$, in the elaborate calculations worked out by Mr. Harbord, on his mysterious value of π.

These facts are incontrovertible, but it it does not follow, that the square root of π is a commensurable quantity; and yet, either with $\pi =$ anything intermediate between 3 and 4; or with $\pi = 3$ with any number of *orthodox* decimals we may be pleased to add, we get the proportion :—Final ratio: ratio 1·25 : : π : 1.

Proof:

$\sqrt{3 \cdot 125} = 1 \cdot 767766 \ldots.$

Then:

```
        1·767766
              ·5 = radius of a circle of diameter 1.
        ·8838830 = side of square.
        ·8838830
       265164900
      70710640
     70710640
    26516490
   70710640
  70710640
 ·781249157689OO = ¼ (π) approximately.
               4 = perimeter of a circumscribing
                   square to a circle of diameter 1.
 3·124996630756OO = π approximately.
             1·25 = 5 (semi-radius) to a circle of
                    diameter 1.
  15624983153780OO
  624999326151200
 312499663075600
 3·9062457884450000
```

Hence: 3·9062457788445 : 1·25 : : 3·1249966630756 : 1;

and $\frac{3 \cdot 9062457788445}{1 \cdot 25}$ and $\frac{3 \cdot 1249966630756}{1}$ are equivalent

ratios. Will any Mathematician *dare* to tell me that these are not equivalent ratios, because $\sqrt{3 \cdot 125}$ is *not* a commensurable quantity? Although 1·767766 is only an approximation to $\sqrt{3 \cdot 125}$, and although, in working out the calculations, other approximate quantities necessarily follow, this does not affect these ratios, and, by extending the number of decimals, we may make the approximations as close as we please, to the following proportion:—

$\{(3 \cdot 125 + \frac{1}{4}(3 \cdot 125))\} : 1 \cdot 25 :: 3 \cdot 125 : 1$; that is,

$3 \cdot 90625 : 1 \cdot 25 :: 3 \cdot 125 : 1$.

Hence: $\frac{3 \cdot 90625}{1 \cdot 25}$ and $\frac{3 \cdot 125}{1}$ are equivalent ratios, and *both* truly express the ratio of circumference to diameter in every circle.

Now, it is self-evident that, we can never find the value of π, either by pure Geometry, or by pure Mathematics. The Sciences of Geometry and Mathematics are *both* involved in the discovery of the true value of π. This I lay down as an axiom, and where is the Mathematician who *dare* dispute it? Now, Geometry and Mathematics are exact Sciences—aye, *the exactest of all Sciences*—and it follows, that what is geometrically true, can never be mathematically false; and conversely, what is mathematically true, can never be geometrically false; in other words, Geometry and Mathematics are the twin handmaids of Science, and never can be inharmonius or inconsistent with each other. Well, then, it would be strange indeed, if the exact Sciences of Geometry and Mathematics were incompetent to teach us how to find the true value of π; and still more strange, if incompetent to furnish us with the means of detecting all *counterfeits*,

however great the names by which these *counterfeits* may have been palmed on the Scientific world.

Now, my dear Sir, I will give *you* a very simple method of detecting all *counterfeits* from the true π; and I think I can do this by logical reasoning from indisputable data, and in such a way that you will be able to follow me without the least difficulty.

Well, then, π denotes the circumference of a circle of diameter = 1, whatever be its arithmetical value, and we must not lose sight of this fact in our search after π. Now, $\frac{1}{4} = \cdot 25$ = semi-radius of a circle of diameter = 1; therefore, 10 (semi-radius) = 5 (radius); that is, $10 \times \cdot 25 = 5 \times \cdot 5 = 2 \cdot 5$. Well, then, let $\sqrt{2 \cdot 5}$ represent the diameter of a circle. Then: $\sqrt{(\cdot 5 + \frac{1}{4} \cdot 5)} = \frac{1}{2}(\sqrt{2 \cdot 5}) = \sqrt{\cdot 625}$ = radius of the circle; therefore, $\sqrt{2 \cdot 5^2} \times \frac{\pi}{4} = \pi \; (\sqrt{\cdot 625^2})$, and this equation = area of the circle, whatever be the value of π; and π is unkown. Now, the value of π must be intermediate between the perimeter of an inscribed regular hexagon, and the perimeter of a circumscribing square, to a circle of diameter = 1; that is, intermediate between 3 and 4. Well, then, if the diameter of a circle = 1, then, 100 (semi-radius) = 25, and $\frac{25}{8} = 5 \; (\cdot 625) = 3 \cdot 125$; and $3 \cdot 125$ is intermediate between 3 and 4; therefore, $3 \cdot 125$ MAY be the value of π. You will observe that this statement is merely inferential, not hypothetical. Now, my dear Sir, if from these facts we can get an equation, connect this equation with a value of π, and prove that no such equation can be obtained by any other value of π, *I appeal to you*, my dear Sir, what is the legitimate conclusion? Well, then, there is a value of π, by which we get the following equation: $5 \; (\sqrt{2 \cdot 5^2} \times \frac{\pi}{4}) = \pi^2$, and by no other

value of π can we get this equation. You ask me:—What is that value of π? *I answer! It is that value of π which makes 8 circumferences = 25 diameters in every circle.*

Proof: $\frac{25}{8} = 3{\cdot}125 = \pi$, therefore, $5\,(\sqrt{2{\cdot}5^2} \times \frac{\pi}{4}) = 3{\cdot}125^2$; that is, $5\,(2{\cdot}5 \times {\cdot}78125) = 9{\cdot}765625$; and since by no other value of π, can we get this equation, it follows that all other arithmetical values of the symbol π but $3{\cdot}125$, are *base counterfeits.*

But we have other methods of detecting and distinguishing all *counterfeits*, from the true value of π.

Let $s\,r$ denote the semi-radius of a circle of diameter $= 1$. Let d denote the diameter, and r the radius of a circle of diameter $= \sqrt{10\,(s\,r)} = \sqrt{2{\cdot}5}$. Then: $\{10(s\,r) \times r^2\} = \frac{\pi}{2}$; that is, $(2{\cdot}5 \times {\cdot}625) = \frac{3{\cdot}125}{2}$; and no other value of π but $3{\cdot}125$ will give this equation.

Again: 5 times the cube of $r = \frac{1}{5}(\pi^2)$; that is, $5\,(\sqrt{{\cdot}625}^{\,3}) = \frac{\pi^2}{5}$, or, $5\,({\cdot}625^2) = \frac{9{\cdot}765625}{5}$; and no other value of π but $3{\cdot}125$ will give this equation.

Again: The square root of $\cdot625$ is an incommensurable quantity. But, the square root of 10 times $\cdot625 = \sqrt{6{\cdot}25} = 2{\cdot}5$. Hence: $\sqrt{\{2\,\pi\,(2{\cdot}5^4)\}} = 5\,(\pi)$; that is, $\sqrt{(244{\cdot}140625)} = 15{\cdot}625$; and no other value of π but $3{\cdot}125$ will give these equations.

Now, my dear Sir, pray try the Orthodox, or any other π, by these tests, and you will find them all wanting; or in other words, *base counterfeits.*

Well, then, where is the Mathematician, who, in the face of these facts, will *dare* to tell me that $3{\cdot}125$ is not

the true arithmetical value of π? Where is the Mathematician, who, in the face of these facts, will *dare* to dispute the truth of the THEORY, *that* 8 *circumferences* = 25 *diameters in every circle?*

I had written so far when your favour of the 21st inst. came to hand, enclosing Mr. R——'s rejoinder to my Letter of the 12th inst. You observe:—"*My relative, Mr. R——, seems to be as stiff as Professor de Morgan.*" True! In certain respects he beats de Morgan far away. The foregoing might have been penned in anticipation of Mr. R——'s last Paper. Let that gentleman "*read, mark, learn, and inwardly digest,*" what I have written. He still declines to admit that π cannot be less than $3\frac{1}{8}$, or so great as $3\frac{1}{8}$; but, this is a matter of little consequence, as he will find that I have now adopted his own limits; that is, that π cannot be greater than 4 nor less than 3. He says:—"*It is something like an insult to say that one lacks candour, when one refuses to admit a petitio principii as a demonstration.*" If Mr. R—— does not "*lack candour,*" let him prove that the demonstrations I herewith submit for his consideration, "*are most remarkable cases of reasoning in a circle,*" and nothing better than a mere "*petitio principii.*" Enough for the present, as regards the last remarkable Paper of Mr. R——.

In his Paper in reply to my Letter of the 22nd February, Mr. R—— observes:—"*Mathematics admit of no assumed theories.*" Indeed, Mr. R——! Where is your proof? Surely so remarkable an assertion requires proof!

Permit me to remind *you*, my dear Sir, of an important remark of Bacon:—"*Theoriarum vires, arcta et quasi se mutuo sustinente partium adaptatione, quâ,*

quasi in orbem cohærent firmantur." * What of this says the Mathematician? Mathematicians know Baconian philosophy to be a *myth*, Bacon himself to have been a *fool*, and all the world gone "*wud*" for putting faith in any such trash and nonsense as proceeded from the pen of Bacon!! Now, my dear Sir, is it not a fair inference that such is Mr. R——'s opinion, when he makes the *bold* assertion, that "*Mathematics admit of no assumed theories?*" If the inference be false, let Mr. R—— prove it! Well, then, I do adopt a THEORY, and can prove the truth of it by practical geometry, fulfilling the conditions of my quotation from Bacon in a thousand ways.

Is Mr. R—— a practical Geometer? If so, let him, from an equilateral triangle, construct a regular octagon within a circle, and I will shew him what results from it. If he does not "*lack candour*," he will surely either comply with my request, or admit his inability to do so.

Mrs. Smith joins me in kind regards to you and yours.

Believe me, my dear Sir,

Very truly yours,

J—— S——, Esq. JAMES SMITH.

Mr. R——'S PAPER, 20*th March*, 1868.

Hitherto I have not had time to write a note or two on Mr. Smith's Paper of the 12th. In it he says that he had in a Letter to Mr. J. S——, proved π determinate; but on looking into that Paper I find no such proof. It contains copy of a Letter to Mr. G——, almost word for word the same as one he sent me

* The confirmation of theories relies on the compact adaptation of their parts, by which, like those of an arch or dome, they mutually sustain each other, and form a coherent whole.

with the same diagram; and which contains the same vitiating assumption that runs through all Mr. Smith's attempts to show that $\pi = 3\frac{1}{8}$.

Mr. Smith complains that I "*catch*" at statements of his. This is true, but not as he insinuates. I try to lay my finger on what is essential to a proof; reducing it to its first principles. Now in this way I have shown that the first principles, the ultimate facts, necessary to establish Mr. Smith's view are not self-evident; that he assumes them without proof: and that he must *prove them* to *be facts* before a sane person can admit their truth.

Now, I again assert that unless Mr. Smith proves those premisses that are involved in the process by which he himself says that he arrived at his value of π, he has no right to claim assent from any one; and certainly will never obtain it from a single Mathematician.

Furthermore, he has not removed the difficulty presented by the fact that there are *several* determinate quantities between the limits of 3 and 4 that fulfil the conditions he lays down. It is easy to prove that 3 and 4 are the limits: but Mr. Smith has not established other limits within these. It is legitimate to attempt this; and I shall be able to understand, and have candour to admit, a satisfactory proof. Only let the *whole* premisses be fairly and clearly proved; because if there be any defect, the conclusion simply does not follow. But I have so frequently and plainly pointed out this, and asked the proofs, in vain, that I am very *nearly* convinced that they do not exist. If Mr. Smith suceed in thus proving that he squares the circle, I shall look at his doctrine of the sines; but the two matters are not shown to have any essential connection with one another, and should be kept separate. If he once prove $\pi = 3\frac{1}{8}$, this fact will go far enough all around.

I now complain of one thing in Mr. Smith's Letter of 25th February, to Mr. J. S——. He there gives his *rule* that if 360 be divided into n parts then $\frac{24}{25}\left(\frac{360}{n}\right) n$ is a constant quantity,

and equal to perimeter of a regular hexagon; and he says that he has not found *one* Mathematician who dares dispute *this fact*, and never one "who has had the candour to admit it"—not excepting Mr. R——. I complain of this.

First: There are TWO facts (so called) in the "RULE" given by Mr. Smith. The one fact is that $\frac{24}{25}\left(\frac{360}{n}\right) n$ is a constant. Who ever questioned this? Where is the boy who has got a smattering of algebra who would dispute it?

The other fact (so called) is that this constant = the perimeter of a regular hexagon. This I HAVE DARED to DISPUTE. Surely Mr. Smith had forgotten this. I called it *no fact* until he had proved *a priori* that it was true. It is something like an insult to say that one lacks candour, when one refuses to admit a *petitio principii* as a demonstration. Mr. Smith's attempt to convince me was the most remarkable case of reasoning in a circle I have ever seen. It arose out of my examination of the Buccleuch demonstration. This "rule" was brought in to establish the premiss of that proof, a proof mark, from which we were to find π. But this very link in the short chain did itself hang on Mr. Smith's π. The π hung on it, and it hung on the π, and not on an independent demonstration. Then, after I objected to that, Mr. Smith revealed his *whole process* of discovery, which again was reduced to several assumptions. (First) π is determinate: (Second) $3\frac{1}{8}$ is the only possible quantity—a quantity which he has fenced round by repeated and arbitrary conditions.

Surely, I may now rest satisfied, that Mr. Smith cannot *prove* $\pi = 3\frac{1}{8}$. His diagrams are interesting, and all these results would be interesting exercises in Geometry were he only able to prove $\pi = 3\frac{1}{8}$. Poor Mr. T—— is sublimely clever. He proves that $\pi = 3$: that, $\therefore$, a part is = to the whole; and that a straight line between two points is equal to the arc of a circle passing through the same 2 points.

MR. JAMES SMITH, *to* J—— S——, ESQ.

BARKELEY HOUSE, SEAFORTH,
26*th March*, 1868.

DEAR SIR,

My last Letter concluded with the following paragraph: "*Is Mr. R—— a practical Geometer? If so, let him, from an equilateral triangle, construct a regular octagon within a circle, and I will show him what results from it. If he does not 'lack candour,' he will surely either comply with my request, or admit his inability to do so.*" Now, my dear Sir, I venture to prophecy, that, he will not construct the required geometrical figure; neither will he admit his inability to do so. If I am a false prophet so much the better for your kinsman; but, even then, the game between us would only be equal, and we should have to think of what must be the next move. Well, then, I will make a move by anticipation.

The following furnishes a description of my method of constructing the enclosed diagram. *(See Diagram V.)*

On the given finite straight line (A B) describe the equilateral triangle O A B. Euclid, Prop. 1, Book 1. This is the very first problem in pure Geometry. Then: with O as centre, and O A or O B as interval, describe the circle X. From the angle B draw the straight line B C at right-angles to O B, and therefore, tangental to the circle X, making B C $=$ $\frac{3}{4}$(O B), and join O C, producing the right-angled triangle, O B C, of which the sides that contain the right-angle, are in the ratio of 3 to 4. With O as centre and O C as interval, describe the circle Y. From the angle C draw the straight line C F at right-

angles to O C, and therefore, tangental to the circle Y, making C F = $\frac{3}{4}$(O C), and join O F, producing the right-angled triangle O C F, of which the sides that contain the right-angle, are in the ratio of 3 to 4. Then: The line C B is perpendicular to the hypothenuse of the right-angled triangle O C F, that is, perpendicular to the line O F. Hence: The triangles O B C and C B F are similar right-angled triangles, and similar to the whole triangle O C F. Euclid, Prop. 8, Book 6. Then: With C as centre and C O as interval, describe an arc to meet the line F C produced at the point D, and join O D, producing the right-angled isosceles triangle O C D. With O as centre and O D as interval, describe the circle Z. Produce the line O C to meet the circumference of the circle Z at the point E, and join D E. Then: D E is a side of a regular inscribed octagon to the circle Z. On O C describe the square O C D G. Produce the sides C O, D G, and O G, to meet and terminate in the circumference of the circle Z at the points L, M, and N, and join L M, M N, and N D. Then: L M, M N, N D, and D E, are sides of a regular inscribed octagon to the circle Z, and any first class school-boy will be competent to complete the figure, and exhibit the inscribed regular octagon to the circle Z.

Now, my dear Sir, permit me to ask you as an *especial* favour to keep this in your *own* possession until you receive Mr. R——'s reply to my Letter of the 19th instant, and for this reason. I look upon you, my dear Sir, as the referee in my Mathematical contest with your kinsman. Now, when you get Mr. R——'s rejoinder to my last communication, you can compare it with this, and so readily convince yourself on the following points.

First: Whether I am a true or a false prophet! Second: Whether your kinsman is, or is not, a fair and candid controversialist!

If I am wrong, his next will give you a copy of the enclosed diagram, or something like it. If he decline to furnish the proof that he is a practical Geometer, you will then see that I am not pitted against a candid antagonist. Whatever course Mr. R—— may pursue, it is my intention to bring under your notice some most interesting and remarkable Mathematical and Geometrical truths, by means of some very simple additions to the diagram.

We hope Mrs. S—— has quite recovered from her recent illness, and with our united kind regards,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

MR. R——'S PAPER, *27th March.*

I have carefully read the Paper sent by Mr. Smith, and were it not that it might seem indicative of a want of respect for labours so voluminous, I would not see it necessary to say anything in reply. He referred me to a Paper of 9th March, for a proof that π is determinate, where there was not a *vestige* of such proof. In this new Paper, you have two facts given as the premisses of a Syllogism, and are told that the conclusion that π is determinate follows from them of necessity; and that π is determinate is incontrovertible; "but it is a thing Mathematicians do not *wish* to see," with other statements

nearly equivalent to calling names; like another specimen of the same kind, where he says that no *honest* Mathematician can dispute his value of π. These things are not reasoning, nor are they soft words that break the bones.

But let us look at this *new* proof that π is a determinate arithmetical quantity. The two facts are of immense importance, teaching us, Mr. Smith says, more than recognised Mathematicians ever dreamed of.

First. If in a circle of diameter $= 1$, c represent the circumference, and a the area, then $c + a = \frac{10\,a}{2}$; that is, $c = 4a$, for these are identical. The opposite sides of this equation are represented by the same arithmetical expression. This is the first fact; nothing new or astonishing either!

Second. The term "*final ratio*" is used here from Mr. H——, in whose calculation it simply means $\frac{5}{4}\pi$; *i.e.*, $\frac{5}{4}$ of the circumference of circle whose diameter $= 1$. And it is stated that if $\sqrt{2 \cdot 5} =$ diameter, then the area $= \frac{1}{2}$ of the "*final ratio*;" or $(\sqrt{2 \cdot 5})^2 = \frac{\pi}{4} = \frac{5}{8}\pi$, which of course is true. All this is seen without the *calculations* used to prove it.

"Now," says Mr. Smith, "if we make the first the major, and the second the minor premiss of a syllogism, the third follows of necessity, viz., that π *must be* a determinate arithmetical quantity.

The first proposition or fact, viz., that if diameter $= 1$, area $= \frac{1}{4}$ circumference, is true; but it is what is called a *particular* proposition. But why analyze the so-called syllogism? Is it really seriously put forth as one? The *predicate of the conclusion* does *not appear at all in the premisses.* First and second are both true; but the conclusion does not follow from them. IT may be true also; but it is not proved to be so by these two propositions. It seems almost disrespectful to Mr. Smith to say all this; but if he will look at the three facts or propositions, he will see that the conclusion asserts what does not follow, and is not contained in the premisses.

If diameter $= 1$, $c + a = \frac{10a}{2}$; or identically.

If diameter $= 1$, area $= \frac{1}{4}$ circumference.

If diameter $= \sqrt{2\frac{1}{2}}$, area $= \frac{5}{8}\pi$. Or, in order to introduce π, the *subject* of the conclusion, into both premisses clearly, put it thus, also identically—

If diameter $= 1$, $\pi r^2 = \frac{1}{2}\pi r$; or, more simply, and still identically—

If diameter $= 1$, π (area) $= \frac{1}{4}\pi$ (the circumference).

If diameter $= \sqrt{2 \cdot 5}$, area $= \frac{5}{8}\pi$. Manipulate them by substituting identical forms, as much as you like, you cannot make a syllogism out of them; for that π is a determinate arithmetical quantity cannot be seen to be their necessary conclusion. In a syllogism, the *predicate* of the *conclusion—i.e.*, the *very thing* you wish to prove about the *subject* of it—must appear, and appear, too, in your *major premiss*. Where is it in either what Mr. Smith calls his major, or in what he calls his minor?

If I take all this trouble in analysing Mr. Smith's Paper, I do certainly beat Professor de Morgan far away in patience and courtesy!

I am perfectly willing to let Mr. Smith have the last word; but if I abstain from further criticism, it will be because I see nothing requiring it. If ever a proof turns up I shall welcome it. Bacon and I do not conflict. *Natural Science* has its *theories;* Mathematics has *theorems*, but no theories; everything must be rigorously *demonstrated.*

As to the problem set me for solution: my time for school and college jokes is past. It is hardly respectful to me to insert these things into a discussion on such a point. It is to play or to throw dust. I do not fear any problem in practical Geometry; but I will not meet such a demand in order to show that I am not unable.

Further, it might be of use if Mr. Smith's Mathematical friend, who *privately* admitted that he had loosed the knot, would

admit it ***publicly***, say in the ***Athenæum***, or any other paper; and give his understanding of the matter.

It is not necessary to criticise Mr. H——. He is too wise to enter *into particulars*, and show how he found his π. He does not *say* that it is determinate. And, in the absence of further information, I would suppose that Mr. H—— found his π by the ordinary method, carrying the calculation out to that enormous length with self-satisfying accuracy.

MR. JAMES SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH,
30th March, 1868.

MY DEAR SIR,

I am in receipt of your esteemed favour of the 28th, enclosing Mr. R——'s remarks on my Letter of the 19th inst., for which please accept my best thanks. I was obliged to be in Manchester yesterday, and consequently could not reply before to-day. You observe:—"*Never say die,*" *seems to be your motto as well as my relative's;* and when truth is the object contended for, you obviously think it a very good motto, for you add:—"*Magna est veritas, et prevalebit, in the end;*" and so, we must "*at it again.*"

In my last to you, I ventured to predict, that Mr. R—— would neither attempt to solve the problem of constructing a regular octagon within a circle from a given equilateral triangle, nor admit his inability to do so. Was I not right in my prediction? How does Mr. R—— excuse himself? By saying:—"***As to the problem set me for solution: my time for school and college jokes is***

past. It is hardly respectful to me to insert these things into a discussion on such a point. It is to play or to throw dust. I do not fear any problem in practical geometry; but I need not meet such a demand in order to show that I am not unable." In this way Mr. R—— fancies he gets over the difficulty, but he will find himself mistaken. When I attempted to bring Mr. R—— to book, by asking him to solve the problem, I also gave him to understand that certain results follow from its solution, and promised to shew him those results. Now, as to these results, it is impossible that Mr. R—— could know anything at the time of writing his Paper, unless he had made a special study of the diagram enclosed in my last Letter to you. *(See Diagram V.)* I appeal to you, my dear Sir, is it "*respectful to me,*" to call my asking him to solve such a problem a "joke"? Is it "*respectful to me*" to say "*it is to play or to throw dust?*" Should not Mr. R—— rather have solved the problem, than make such assertions? Would he not have done so, had he been competent to the task? Had he solved the problem, and by means of it established the infallibility of orthodoxy, would he not have proved, that I am a mere *empiric*, vainly attempting "*to throw dust*" into the eyes of a real mathematician? But, never mind, I have had to submit to be so treated over and over again. I may tell you, my dear Sir, that I have not been in constant correspondence with some mathematical *savan* or other for the last 7 or 8 years, without becoming perfectly acquainted with the ways of our recognised mathematical authorities; indeed, I venture to say—even at the risk of being thought egotistical—that I know them better than they know themselves.

Well, then, you *now know*, my dear Sir, that the solution of the problem of constructing a regular inscribed octagon to a circle, from a given equilateral triangle, is *un fait accompli:* proved beyond the possibility of dispute, by the enclosed diagram. *(See Diagram V.)* Will Mr. R—— venture to say, that he knew how to solve this problem, until I shewed him the way? But, never mind! Now for some of the results that follow from the solution of this problem.

Let X, Y and Z denote the areas of the three circles. Then: $3\frac{1}{8}$ (X) = Z, whatever be the value of π.

Proof: Let O B the radius of the circle X = 4. Then: O C the radius of the circle Y = 5; and O D the radius of the circle Z = $\sqrt{50}$, by construction: and, because π (O B^2) = X; π (O C^2) = Y, and π (O D^2) = Z, whatever be the value of π; it follows of necessity, that we may prove by any *determinate* value of π intermediate between 3 and 4, that $3\frac{1}{8}$ (X) = Z. For example: By hypothesis, let $\pi = 3{\cdot}1416$, a very close approximation to its true value on the orthodox theory, and orthodoxy has its theories whatever Mr. R—— may be pleased to say to the contrary, or in other words, whatever distinction may be *floating* in his mind between *theorems* and *theories* in Mathematics. Then: π (O B^2) = $3.1416 \times 4^2 = 3{\cdot}1416 \times 16 = 50{\cdot}2656$ = X; and π (O D^2) = $3{\cdot}1416 \times \sqrt{50}^2 = 3{\cdot}1416 \times 50 = 157{\cdot}08$ = Z, that is, $3\frac{1}{8}$ $(50{\cdot}2656) = 157{\cdot}08$; and, any other *finite hypothetical* value of π, intermediate between 3 and 4, will produce a similar result. Will any Mathematician *dare* to tell me that the two sides of the equation $3\frac{1}{8}$ (X) = Z, are unequal? I trow not! Well then, this fact—even if it stood alone, without confirmation of any kind—

"*upsets*" the orthodox notion, that π is an ***indeterminate*** quantity, and only arithmetically expressible by an infinite series. A word *en passant.* The fluxionary calculus of Newton, and the differential and integral calculus of Leibnitz, *(are not both theories?)* have done far more to retard the advancement of valuable scientific knowledge, than they have ever done to promote it. Well, then, Mr. R—— will not attempt to *upset* my facts, but he may probably tell us, that I am playing with *him*, and attempting "*to throw dust*" into *your* eyes.

Now, Mr. R—— might very fairly put the question:— How do your facts prove $\pi = 3\frac{1}{8}$? I answer, in the following way: $3\frac{1}{8}$ (O B^2 + B C^2 + O C^2) = $12{\cdot}5\left(\frac{\text{O D}}{2}\right)^2$; that is, $3\frac{1}{8}$ times the sum of the areas of squares about the triangle O B C = Z. Hence: The equation $3\frac{1}{8}$ (O B^2 + B C^2 + O C^2) = $12{\cdot}5\left(\frac{\text{O D}}{2}\right)^2$ = area of the circle Z. Pray, Mr. R——, find another value of π by which you can produce this result; or in other words, find some other value of π by which you can get this equation. I appeal to *you*, my dear Sir, is it disrespectful, unfair, or unreasonable, that I should ask this of Mr. R——? Now, I venture to predict, that Mr. R—— will either treat my arguments and conclusion with silent contempt; or, boldly assert that they prove nothing. I am disposed to think the former the more probable, and that in this way he will put an end to our controversy, by letting me have "*the last word.*"

I have given you in a former Letter one method, and will now give *you*, my dear Sir, in connection with the diagram, another very simple method of detecting the true π, from all counterfeits.

Well, then, $4\ (O\,D^2) = 16 \left(\frac{O\,D}{2}\right)^2$, and this equation = area of a circumscribing square to the circle Z. Now, let A denote the area of a square circumscribed about the circle Z. Let M denote $\left(\frac{O\,D}{2}\right)^2$. Let N denote $\frac{M}{2}$. And let P denote $\frac{2}{3}$ (M) = $\frac{4}{3}$ (N). Then: we get the following algebraical formula for finding the true arithmetical value of π, and distinguishing it from all counterfeits.

$$\frac{A}{16} = M\,;\ \frac{M}{2} = N\,;\ \frac{2}{3}(M) = \frac{4}{3}(N) = P \therefore 24\,(P) = A.$$

Proof:

Let A be represented by any arithmetical quantity, say 666, and denote a circumscribing square to the circle Z.

Then:

$$\frac{A}{16} = \frac{666}{16} = 41{\cdot}625 = M.$$

$$\frac{M}{2} = 41{\cdot}625 = 20{\cdot}8125 = N.$$

$$\frac{2}{3}(M) = \frac{4}{3}(N) = \frac{83{\cdot}25}{3} = 27{\cdot}75 = P.$$

$$\therefore 24\,(P) = 24 \times 27{\cdot}75 = 666 = A.$$

Hence:

$$10\,M + 5\,N = 10\,M + \tfrac{1}{4}(10\,M) = 12{\cdot}5\,(M).$$

$$\therefore \frac{10\,M + 5\,N}{\frac{1}{4}\pi} = \frac{10\,M + \frac{1}{4}(10\,M)}{\frac{1}{4}\pi} = \frac{12{\cdot}5\,(M)}{\frac{1}{4}\pi} = A.$$

$\therefore \frac{12{\cdot}5}{4} = 3{\cdot}125 = \pi.$ $\frac{12{\cdot}5\,(M)}{\frac{1}{4}\pi} = 24$ (P), and this equation = area of a circumscribing square to the circle Z. But, M = area of a square on the semi-radius of the circle Z, and since the property of one circle is the property of all circles, it follows of necessity, that $12\frac{1}{2}$

times the area of a square on the semi-radius = area in every circle.

Well then, if you can, *induce* Mr. R—— to find an algebraical formula, and connect it with some simple geometrical figure such as the enclosed. If he meet this by saying:—"*I need not meet such a demand in order to shew that I am not unable*," what, my dear sir, will you, as referee, think of Mr. R——?

You will very likely wonder why I selected the number 666 for my demonstration. I will tell you why. If you refer to the *Athenæum* of the 27th October, 1866, you will find that Professor de Morgan there introduces into his Budget as a paradoxer, my late esteemed friend, the Rev. David Thom, (and I am coupled with him in the same article,) under whose ministry I sat for upwards of 30 years. The Dr. wrote a remarkable work, entitled: "*The Number and Names of the Apocalyptic Beasts;*" and the Professor's attack has special reference to this work. This circumstance occurred to me while writing, and I adopted these figures as they suited my purpose quite as well as any others. For years before his death Dr. Thom was stone blind, but after becoming blind he continued to attend to his ministerial duties for some time. It so happened that after his decease, I edited a volume of the Doctor's sermons, taken down in short hand, and wrote an address to the reader, signed, simply, the Editor. On reading the article referred to, I sent de Morgan a copy of the volume of sermons, writing my own name under the signature to the address to the reader, which drew from the Professor a direct communication, of which the following is a copy:

"Mr. de Morgan *presents his kind regards to* Mr.

James Smith, *and, with all the astonishment which he has no doubt was anticipated, returns his best thanks for the copy of Mr. Thom's Sermons presented to him by* Mr. James Smith *as Editor, and his best wishes in exchange for those of* Mr. Smith.

91, Adelaide Road,
October 29, 1866."

This is the only note I ever received from the Professor, and his hand-writing was only previously known to me, from the direction on an envelope, at the time we exchanged photographs.* I think this little episode will amuse you.

Enough for the present. Let us wait and see what Mr. R—— will make of this communication, and then some more very remarkable results shall follow.

Mrs. Smith is glad to hear Mrs. S—— continues to improve, and with our united kind regards,

Believe me, my dear Sir,
Very faithfully yours,
JAMES SMITH.

J—— S——, Esq.

* See *Athenæum*, July 29, 1865; Article, "Budget of Paradoxes." A. de Morgan.

Mr. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
1st April, 1868.

MY DEAR SIR,

Your relative, Mr. R——, commenced his last Paper by observing:—"*I have carefully read the Paper sent by Mr. Smith, and were it not that it might seem indicative of a want of respect for labours so voluminous, I would not see it necessary to say anything in reply:*" and, in another part, he says:—"*I am perfectly willing to let Mr. Smith have the last word, but if I abstain from further criticism, it will be because I see nothing requiring it. If ever a proof turns up I shall welcome it.*" Now, my dear Sir, you will know by this time that I have arrived at the conclusion, that no proof ever can ***turn up*** that will convince Mr. R——. De Morgan can write truth occasionally, and in one number of his "Budget of Paradoxes" he said:—"*Crammed erudition seldom casts out any hooks for more*"—and he never gave utterance to a greater truism. So true is it, that, I may say without hesitation, our great Mathematicians are carried by "*crammed erudition*" so far beyond the regions of common sense, that no reasoning on Geometry and Mathematics, however sound and logical, if not in accordance with what they have been taught, can ever find an entrance into their minds. Well, then, Mr. R—— may probably "*abstain from further criticism,*" *not seeing anything requiring it;* at any rate, I cannot have another Paper of his in reply to mine of yesterday before next week, and, by way of episode, I will in the mean time give you a copy of an epistle which I think will amuse you.

When I published my Letter to the Duke of Buccleuch, on the Quadrature and Rectification of the Circle, it was honoured with a notice in the leading Scientific Journal, which you will find in the *Athenæum*, of the 14th September, 1867.* I was induced, in consequence, to address a Letter to the Editor, of which the following is a copy:—

* FROM THE "ATHENÆUM" OF 14TH SEPTEMBER, 1867.

Letter to his Grace the Duke of Buccleuch, on the Quadrature and Rectification of the Circle. By James Smith, Esq. (Simpkin, Marshall & Co.)

Ecce iterum Crispinus, says the Latin poet; here we are again, says the clown; May it please your Grace, says Mr. Smith. And it ought to do so; for the phrases are all fashioned upon the hypothesis that the Duke will read and understand, which is a compliment to his industry and intelligence that any duke in Christendom might be proud of. His Grace will indeed be the "bold Buccleuch" if he does the first and says the second. The circle is the old emblem of eternity; the symbol refers to the efforts to square it. The sub-divisions of eternity are times. This time Mr. Smith brings on the stage the Rev. G. B. Gibbons, between whom and the $3\frac{1}{8}$-ist upwards of 120 letters have passed. We hope this means only six tens from one and half-a-dozen tens from the other; not 120 letters a piece. Mr. Gibbons is the second Mathematician whom Mr. James Smith has whiled into a long correspondence with him. Mr. de Morgan—who of course is handsomely acknowledged—showed his sense by never giving a private answer to any one of Mr. Smith's private letters. He knew that Mr. Smith is the Old Man of the Sea in the Arabian Nights, who would not dismount from the neck of anyone who let him get up for a ride. Journals cannot be so served; and *we* are glad to meet Mr. Smith again. We hope to have a bit of sport with him many a time in the future, as we have had in the past. He is now an institution; and here we go round, round, round, (and $\frac{1}{8}$ of a round, of course) is a regular part of our itinerary. As to the rest, there are two sines to an angle, geometrical and trigonometrical; one of them, no matter which, is greater than the measure of the angle in small angles; the mathematicians are a set of priests, who jealously guard a mystery;

14th September, 1867.

To the EDITOR OF THE "ATHENÆUM."

SIR,

I beg to thank you for the complimentary notice of my Letter to his Grace the Duke of Buccleuch, on the Quadrature and Rectification of the Circle, which appears in the *Athenæum* of to-day. Complimentary it is, although you did not so intend it, and I ground this expression of opinion upon the following facts:—Assertion is not demonstration, neither is ridicule argument; and since you have not dared to point out any inadmissible step in my data, or attempt to grapple with any of my demonstrations by logarithms, you have left my arguments unanswered, and my conclusions uncontradicted. Now, Sir, there is far more in Geometry than was ever dreamed of in your mathematical philosophy, and I take the liberty of directing your attention to some geometrical and mathematical truths, which will be perfectly new to you, and I shall make use of them to put your honesty and sincerity, as a Scientific Journalist, to the test!

The following may be taken as the method of constructing the enclosed diagram. (*See Diagram VI.*)

From a given straight line (A B) construct a right-angled triangle A B C, making A B and B C the sides that contain the right angle, in the ratio of 4 to 3. With A as centre, and A B as interval, describe the circle X, and with B as centre, and B A as interval, describe the circle X′. Join the points D and E where the circumferences of the circles X and X′ cut each other. Produce

and Albert the Good will ever be revered, for when Mr. Smith sent one of his books, the Prince's librarian returned thanks for "the valuable addition made to His Royal Highness's Library." *(Our Library Table.)*

BA to meet the circumference of the circle X at the point F, and join FD and FE. Then, FDE is an inscribed equilateral triangle to the circle X, and the lines AB and DE intersect and bisect each other at the point O; therefore, the line FO which bisects the equilateral triangle FDE = 3 times AO or OB = 2(BC); therefore, AO + ½ (OB) = BC = AG. With A as centre and AC as interval, describe the circle Y: with A as centre and BC as interval, describe the circle Z: and, with A as centre and DE as interval, describe the circle XZ. The circumference of the circle XZ cuts the circumference of the circle X′ at two points, and it is self-evident that if straight lines be drawn from the point A the centre of the circle X to these points, these lines would be sides of an inscribed equilateral triangle to the circle X′, and radii of the circle XZ.

Now, let AB the radius of the circles X and X′ = 4, which makes BC and AG = 3, by construction.

Then:

π (BC^2) or π (AG^2) = π (3^2) = area of the circle Z.

π (AB^2) = π (4^2) = area of the circle X.

π (AC^2) = π (5^2) = area of the circle Y.

π (DE^2) = π ($\sqrt{48^2}$) = area of the circle XZ.

$3\ (AB^2) = DE^2$.

$BC : AB :: DE^2 : FB^2$.

$BC^2 : AB^2$: : area of circle Z : area of circle X.

And, $BC : AB :: AB : 5\frac{1}{3}$.

Hence:

BC : AB : : the area of a square on DE : the area of a circumscribing square to the circles X and X′. The area of the circle Z : the area of the circle X or X′ : : the area of a square on BC : the area of a square on AB.

The sum of the areas of the circles Z and X = area of the circle Y. And, the area of the circle X Z = $5\frac{1}{3}$ times the area of the circle Z; 3 times the area of the circle X; and 1·92 times the area of the circle Y.

Let the letters which represent the circles, denote the arithmetical values of their areas. Then:

$$X\,Z : Z :: 5\tfrac{1}{3} : \text{unity}.$$
$$X\,Z : X \text{ or } X' :: 3 : \text{unity}.$$
$$X\,Z : Y :: 1\tfrac{92}{100} : \text{unity}.$$

All the foregoing facts are quite independent of the true arithmetical value of π. In other words, for the purpose of ascertaining the relative values of area to area in the circles, we may *hypothetically* adopt 3·1416, 3·14159, or 3·14159265 as the value of π; or we may work out the calculations on any other *hypothetical* value of π greater than 3 and less than 4; and so, vary the arithmetical values of the areas of the circles; but, we cannot alter the ratios or relative values of circle to circle; that is to say, $XZ :: Z :: 5\frac{1}{3} : 1$; $XZ : X$ or $X' :: 3 : 1$, and, $X\,Z : Y :: 1\frac{92}{100} : 1$, by whatever *hypothetical* value of π we may work out the calculations, to get the areas of the circles.

Again: 3 (X Z), 16 (Z), and $4\,\pi\,(F\,O^2)$ are equivalent expressions; and, if these expressions be denoted by P, then, P = area of a circle of which F O the bisecting line of the isosceles triangle F D E is the semi-radius. One *hypothetical* value of π intermediate between 3 and 4, is just as good as another for the purpose of demonstrating these facts.

Again: The following fact is inseparably connected with the geometrical figure represented by the diagram. $(A\,B^2 + B\,C^2 + A\,C^2) = 3{\cdot}125\,(A\,B^2)$, whatever arith-

metical value we may put upon A B the radius of the circles X and X′; that is to say, the sum of the areas of squares about the right-angled triangle A B C, the generating figure of the diagram = 3·125 times the area of a square on A B the perpendicular.

Again: Let n denote the area of the circle Z. Then: $\frac{n}{FO^2} = \frac{\pi}{4}$; and, $\frac{\pi}{4}$ = area of a circle of diameter unity, whatever be the value of π. That the equation $\frac{n}{FO^2} = \frac{\pi}{4}$ may be demonstrated by means of any *hypothetical* value of π, intermediate between 3 and 4. For example: By hypothesis, let $\pi = 3·14159265$; and let A B, the radius of the circles X and X′ = 4. Then: B C the base of the right-angled triangle A B C = $\frac{3}{4}$ (A B) = 3, by construction. But, A G = B C by construction, and A G is the radius of the circle Z; therefore, π (A G²) = π (B C²) = 3·14159265 × 9 = 28·27433385 = n; and A F + A O = F O = 6, when A B = 4; therefore, $\frac{n}{FO^2} = \frac{28·27433385}{36} = \frac{3·14159265}{4·} = ·7853981625 = \frac{\pi}{4}$.

Again: by hypothesis, let $\pi = 3·1416$, and let F B the diameter of the circle X = 1. Then: A B the radius of the circles X and X′ = $\frac{1}{2}$ = ·5; and A G = B C = $\frac{3}{4}$ (A B) = ·375; therefore, π (A G²) = π (B C²) = 3·1416 × ·375² = 3·1416 × ·140625, = ·4417875 = n; therefore, $\frac{n}{FO^2} = \frac{·4417875}{·75^2} = \frac{·4417875}{·5625} = ·7854 = \frac{\pi}{4}$.

Now, my good Sir, any other *hypothetical* value of π, intermediate between 3 and 4, will produce similar results, and if you are incompetent to convince yourself of this fact, you must be more of a "*clown*" than a philosopher.

Well, then, on the THEORY that 8 circumferences of a circle are exactly equal to 25 diameters, which makes $\frac{25}{8}$ $= 3{\cdot}125$, the arithmetical value of π; $(\cdot AB^2 + BC^2 + AC^2)$ $= \frac{n}{FO^2} = \frac{\pi}{4} =$ area of the circle X, when F B the diameter of the circle = unity = 1.

Proof: If F B the diameter of the circle X = 1, A B the radius of the circles X and X′ = ·5, and $\frac{3}{4}$ (A B) = ·375 = B C, the base of the triangle A B C; and B is a right angle, by construction; therefore, $AB^2 + BC^2 = {\cdot}5^2 + {\cdot}375^2 = {\cdot}25 + {\cdot}140625 = {\cdot}390625 = AC^2$; therefore, $(AB^2 + BC^2 + AC^2) = {\cdot}25 + {\cdot}140625 + {\cdot}390625 = {\cdot}78125$ = the sum of the areas of squares on the sides of the right-angled triangle A B C, the generating figure of the diagram $= \frac{\pi}{4} =$ area of a circle of diameter = unity. But, $\pi\,(BC^2) = 3{\cdot}125 \times {\cdot}375^2 = 3{\cdot}125 \times {\cdot}140625 = {\cdot}439453125 = n$; that is, = area of the circle Z; and F O, the bisecting line of the isosceles triangle F D E $= \frac{3}{4}\,(FB) = \frac{3}{4}\,(1) = {\cdot}75$; therefore, $\frac{n}{FO^2} = \frac{{\cdot}439453125}{{\cdot}75^2}$ $= \frac{{\cdot}439453125}{{\cdot}5625} = {\cdot}78125 = \frac{\pi}{4} =$ area of a circle of dia- unity. Hence: We get the equation, $(AB^2 + BC^2 + AC^2)$ $= \frac{\text{area of the circle Z}}{FO^2}$, and this equation is incontrovertible, and holds good whatever arithmetical value we may be pleased to put upon A B the radius of the circles X and X′.

Now, Sir, if you can find any other value of π either greater or less than $3{\cdot}125$, by which you can obtain this equation, you will be something more than a philosopher, and will prove to the Scientific world that you are indeed

a phenomenon, even more remarkable than "*the Old Man of the Sea, in the Arabian Nights.*"

Again, Sir: You will not dispute that mathematical and geometrical data admit of no doubt, and I shall now adopt a datum with regard to which it is impossible you can raise an objection. In every circle, the perimeter of a regular inscribed hexagon = 6 times radius; and $\frac{3}{\pi}$ expresses the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π.

Well, then, if the radius of a circle = 1, the perimeter of a regular inscribed hexagon = 6. Let this be our datum; and if, upon this datum, I stumble into illogical reasoning, it will be for you to prove it, and so vitiate my conclusions.

Now, $\frac{3}{3\cdot125}$ (6) = 5·76, and, $\frac{3\cdot125}{3}$ (6) = 6·25; therefore, 5·76 : 6 : : 6 : 6·25, and the mean proportional between 5·76 and 6·25 is $\sqrt{5\cdot76 \times 6\cdot25} = \sqrt{36} = 6$ = the perimeter of a regular inscribed hexagon to a circle of radius 1. Hence: $\frac{3}{3\cdot125}$ expresses the ratio between the perimeter of every regular hexagon, and the circumference of its circumscribing circle.

Proof: The following is an example of continued proportion:—

$$a : b : : b : c : : c : d : : d : e.$$

Hence: $\frac{3}{3\cdot125}(e)=d$: $\frac{3}{3\cdot125}(d)=c$: $\frac{3}{3\cdot125}(c)=b$: and $\frac{3}{3\cdot125}(b)$ $=a$: and, when $e=6\cdot25$: $d=6$: $c=5\cdot76$: $b=5\cdot5296$: and a $=5\cdot308416$: therefore, $\frac{3\cdot125}{3}(a)=b$: $\frac{3\cdot125}{3}(b)=c$: $\frac{3\cdot125}{3}(c)$ $=d$, and, $\frac{3\cdot125}{3}(d)=e$; and it follows of necessity,

that $b \times d = a \times e$, and c the middle term is a mean proportional between a and e the extreme terms; and is also a mean proportional between b and d. b is a mean proportional between a and c, and d is a mean proportional between c and e.

Therefore:

When $e =$ circumference of a circle, $d =$ perimeter of a regular inscribed hexagon.

When $d =$ circumference of a circle, $c =$ perimeter of a regular inscribed hexagon.

When $c =$ circumference of a circle, $b =$ perimeter of a regular inscribed hexagon.

When $b =$ circumference of a circle, $a =$ perimeter of a regular inscribed hexagon.

Now, Sir, we may put any arithmetical value we please upon e, and in working out the calculations backwards from e to a obtain exact arithmetical results; but if we put an arithmetical value upon a, and attempt to work out the calculations forwards from a to e, (unless we first get a value of a by making the calculations backwards from e to a), we MAY get into incommensurables immediately. For example: If $a = 6{\cdot}25$, then, $\frac{3{\cdot}125}{3}$ $(a) = \frac{3{\cdot}125 \times 6{\cdot}25}{3} = \frac{19{\cdot}53125}{3} = 6{\cdot}510416$ with 3 to infinity. But, will you *dare* to tell me that for this reason the middle term c is not, and cannot be, a mean proportional between the extreme terms of a and e? You would indeed be more of a mathematical "*clown*" than a philosopher, if, unable to perceive that the difficulty, *(if difficulty it can be called)*, arises from the mere fact

that all numbers are not divisible by 3 and its multiplies, without a remainder.

Again, Sir: You will not dispute that $\frac{3}{\pi}$ and $\frac{6}{2\pi}$ are equivalent ratios, and that both express the ratio between the perimeter of every regular hexagon and the circumference of its circumscribing circle, whatever be the value of π; and I must now direct you attention to the following algebraical formula:—

$\frac{2\pi}{6}(a) = b : \frac{2\pi}{6}(b) = c : \frac{2\pi}{6}(c) = d$, and so on, *ad infinitum.*

Hence: If a represent the perimeter of a regular hexagon, b represents the circumference of its circumscribing circle. If b represent the perimeter of a regular hexagon, c represents the circumference of its circumscribing circle. If c represent the perimeter of a regular hexagon, d represents the circumference of its circumscribing circle, and so on, *ad infinitum.* Will you *dare* to dispute the formula, because you cannot put a value on a and work out the calculations from a to d with arithmetical exactness? I trow not! Well, then, to vitiate my conclusion that 8 circumferences of a circle are exactly equal to 25 diameters, which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π, you must not only dispute my data, and prove my reasoning to be illogical, but you must find an arithmetical expression for the ratio between the perimeter of a regular hexagon and the circumference of its circumscribing circle, by which you can put a value on d, work out the calculations backwards to a, and prove that b is a mean proportional between a and c, and c a mean proportional between b and d. Think of "*the Old Man of the Sea, in the Arabian Nights!*"

Again: In the triangle A B C, the generating figure of the diagram, the sides A B and B C are in the ratio 4 to 3, and B a right angle, by construction.

Now, $AB : AC :: AC : m :: m : n :: n : p$. This is an example of continued proportion: and when $AB + \frac{1}{4}(AB) = AC$: $AC + \frac{1}{4}(AC) = m$: $m + \frac{1}{4}(m) = n$; and $n + \frac{1}{4}(n) = p$. Hence: when $AB = 4$, $AC = 5$: $m = 6{\cdot}25 = 2\pi$, $n = 7{\cdot}8125$ = 10 times the area of a circle of diameter unity: and $p = \pi^2 = 3{\cdot}125^2 = 9{\cdot}765625$.

Hence:

$AB \times n = AC \times m = 10(\pi) = 31{\cdot}25$.

$AC \times p = m \times n = 5(\pi^2) = 48{\cdot}828125$.

$AB \times p = AC \times n = (2\pi)^2 = 39{\cdot}0625 = m^2$.

Therefore:

The middle term m is a mean proportional between the extreme terms A B and p, and is also a mean proportional between the terms A C and n. A C is a mean proportional between A B and m, and n is a mean proportional between m and p. Hence: m^2 is a mean proportional between $10\ \pi$ and $5\ (\pi^2)$; that is, $\sqrt{(31{\cdot}25 \times 48{\cdot}828125)} = \sqrt{1525{\cdot}878\overline{90625}} = 39{\cdot}0625 = m^2$.

Again: Let X denote the area of a square, and let Y denote the area of its circumscribing circle.

Then:

$\{(X + \frac{1}{4}X) + \frac{1}{4}(X + \frac{1}{4}X)\}$: $2(X) \times \frac{\pi}{4}$ and: $\left(\frac{\sqrt{2X}}{2}\right)^2 \times \pi$, are equivalent alegbraical expressions, and are all equal to Y.

Thus, from these facts we obtain a method of finding the area of a circle, from the given area of its inscribed square, which shivers to atoms the absurd orthodox *assumption*, that π can only be expressed arithmetically by an infinite series.

Now, Sir, you have charged me with being a fool, or at any rate, you have endorsed the charge, by permitting Mr. de Morgan so to call me repeatedly, in the columns of your Journal. You profess to be, and if you really are, a Mathematician, you will point out a flaw in my data, or a fallacy in my reasonings, and thus vitiate my conclusion, establish the truth of Mr. de Morgan's charge, and put me to silence. If you find this to be impossible, and are an honest scientific Journalist, you will purge your conscience, by admitting my conclusions; and so far as you are concerned, acquit yourself of any participation in Mr. de Morgan's scurrility and ribald vulgarity.

I shall now assume a thing which remains to be proved. In this communication, "*the phrases are all fashioned upon the hypothesis*," that you, Mr. Editor, are a philosopher and a gentleman; and consequently, that "*you will read and understand, which is a compliment to your industry and intelligence.*" Now, if my hypothesis be well founded, you will "*read, mark, learn, and inwardly digest*" this epistle; and doing so, you cannot fail to be thereby translated from the kingdom of mathematical darkness, into the kingdom of geometrical light, and thus become a "$3\frac{1}{8}$-*ist;*" and you will prove to the scientific world, that you are an honest convert to geometrical truth, by making a frank confession of your faith in $3\frac{1}{8}$-*ism,* through the columns of the leading Scientific Journal.

Now, let the area of a square be represented by any arithmetical quantity, and be given to find two pairs of numbers, of which the mean proportional between these two pairs of numbers shall be equal to the given area of the square. For example: Let the given area of the

square be 7. Then: 14 and 3·5 will be the first pair of numbers, and 10·9375 and 4·48 will be the second pair of numbers; therefore, $\sqrt{14 \times 3·5} = \sqrt{10·9375 \times 4·48}$; and this equation $= \sqrt{49} = 7$; therefore, the given area of the square is the mean proportional between the two pairs of numbers.

Again: Let the side of a square be represented by any aritmetical quantity, and be given to find two pairs of numbers, of which the mean proportional between these two pairs of numbers, shall be represented by the same arithmetical symbols. In this case we may work out the calculations, either outwards or inwards. For example: Let the given side of the square be $\sqrt{5}$. Then: If outwards, the first pair of numbers will be 10 and 2·5, and the second pair of numbers will be 7·8125 and 3·2; therefore, $\sqrt{10 \times 2·5} = \sqrt{7·8125 \times 3·2}$, and this equation $= \sqrt{25} = 5$; therefore, the mean proportional between these two pairs of numbers is represented by the same arithmetical symbol. If the calculations be worked out inwards, the first pair of numbers will be 5 and 1·25, and the second pair of numbers will be 3·90625 and 1·6; therefore, $\sqrt{5 \times 1·25} = \sqrt{3·90625 \times 1·6}$, and this equation $= \sqrt{6·25} = 2·5$; therefore, the mean proportional between these two pairs of numbers is represented by the same arithmetical symbols. It is *axiomatic*, that if $\sqrt{5}$ denote the diameter of a circle, $2·5 =$ area of an inscribed square.

Now, Sir, permit me to put the following question. Do you know the *rule* by which these peculiar mean proportionals are found? If you are a mathematical philosopher, and a gentleman, will you not answer this

question, by giving the *rule?* If you do not know the *rule*, and are a gentleman, will you not admit the fact? If you decline to do either, and fancy you are protected by the privileges connected with your editorial capacity, will you not falsify my *hypothesis*, and prove that you are neither a philosopher nor a gentleman? In putting these questions, do I not fulfil the promise made in the first page of this communication, and put your honesty and sincerity as a Scientific Journalist to the test? You say, in your notice of my Letter to the Duke of Buccleuch, that "*you hope to have a bit of sport with me many a time in the future, as you have had in the past.*" Now's your time. This epistle affords you the opportunity. Don't fail to let me share in the "*sport*" you get out of it.

In conclusion: You charge me with having "*whiled*" two Mathematicians into a long correspondence. This is not true! I "*whiled*" neither of the gentlemen to whom you refer into any correspondence at all. Our correspondence was of their seeking, not mine, and both are personally unknown to me at this moment. Do not imagine, my good Sir, that I have any desire to play the part of a *wily* disputant and attempt to *while* you into a long correspondence. But I confess, I should like you to look upon this communication as a last effort to *wile* or *while* you and Professor de Morgan into honest Mathematicians; and if the pair of you have "*the power to see truth and the candour to admit it,*" I shall most assuredly be successful, and little further correspondence will be necessary on my part, with reference to the true ratio of diameter to circumference in a circle.

I am, Sir, yours respectfully,

J——s S——h.

Now, my dear Sir, I will give *you* the rule for finding from a given number, two pairs of numbers, so that the given number shall be the mean proportional between both pairs of numbers. This may be put in the form of a theorem.

THEOREM.

Let n denote any given finite quantity. Find two pairs of numbers, so that n shall be the mean proportional between both pairs of numbers.

The following algebraical formula solves the theorem. $2\,n$ and $\frac{n}{2}$ will be one pair, and $\frac{2\,n}{\pi}$ and $\frac{n}{2} \times \pi$ will be the other pair of numbers. For example: Let $n = 10$. Then: $2\,n = 20$, and $\frac{n}{2} = 5$; and 20 and 5 will be one pair of numbers; $\frac{2\,n}{3\frac{1}{8}} = 6·4$, and $\frac{n}{2} \times 3\frac{1}{8} = 15·625$; and 6·4 and 15·625 will be the other pair of numbers. Hence: $\sqrt{(20 \times 5)} = \sqrt{(6·4 \times 15·625)}$, and this equation $= n$. In other words, n is the mean proportional between the two pairs of numbers, 20 and 5, and ·6·4 and 15·625.

I fancy I hear Mr. R—— exclaim! Is not $3\frac{1}{5}$ a *determinate* number? Do you mean to say, that $3\frac{1}{5}$ will not produce a similar result? Can we not find two pairs of numbers of which n shall be the mean proportional, with $\pi = 3\frac{1}{5}$? In a geometrical and mathemetical enquiry, there is never occasion to *shirk* a fact, and I answer, without any hesitation, *Yes!* For example: $\frac{20}{3·2} = 6·25$, and $\frac{10}{2} \times 3·2 = 16$; therefore, $\sqrt{(20 \times 5)} = \sqrt{(6·25 \times 16)}$, and this equation $= n$. But $3\frac{1}{8}$ and $3\frac{1}{5}$ cannot both be the true value of π, and the value of π remains

to be discovered by other means. Well, then, how are we to find the true value of π, and distinguish it from all *counterfeits?*

Now, my dear Sir, I have not only shewn you how to find the true value of π in many ways, but I have shewn you how you may detect all *counterfeits*; but I know of no better method of distinguishing the true value of π from all *counterfeits*, than that I have given you on page 12 of this communication, (*see p.* 96) in which I have proved that $(2\pi)^2$ is a mean proportional between $10(\pi)$ and $5(\pi^2)$ which can be predicated of no other but the true value of π; that is, of that value of π which makes 8 circumferences = 25 diameters in every circle, making $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π; and this settles the knotty point of the true ratio of diameter to circumference in a circle.

In his last Paper Mr. R—— says: "Mr. Harbord does not make π *determinate.*" How Mr. R—— can have worked up his imagination into the belief of this fancy, is to me a mystery. It is as plain as that 2 and 2 make 4, that Mr. Harbord's *only* ground for asserting that he has squared the circle, is, that he has hit upon figures by which he makes $5(\sqrt{\pi})^2$ *determinate.* The expression $5(\sqrt{\pi})^2$ with Mr. Harbord, denotes the area of a circle of diameter 1, which he calls "*final and conclusive,*" by which he means that π is *determinate*; for, he knows as well as either Mr. R—— or myself, that $\frac{\pi}{4}$ = area of a circle of diameter 1, whatever be the value of π. Now, I have shewn that we may find as many values of π as we please, much more simple than Mr. Harbord's, and make the expression $5(\sqrt{\pi})^2$ a *determinate* quantity.

Well, then, either with $\pi = 3\frac{1}{8}$ or $\pi = 3\frac{1}{5}$, we may find two pairs of numbers, from a given number, so that the given number shall be the mean proportional between both pairs of numbers ; and pray what becomes of Mr. R——'s *determinate* numbers, $3\frac{4}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, and $3\frac{1}{3}$, by which he fancies he "*upsets*" my coach : but I beg to tell him that before he can upset my coach, he must find a value of π, by which he can make $(2\pi)^2$ a mean proportional between $10(\pi)$ and $5(\pi^2)$. Let him try ! He will find that $3\frac{1}{8}$ will not do it, much less $3\frac{4}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, and $3\frac{1}{3}$!!

I followed up my Letter to the Editor of the *Athenæum* by another, of which the following is a copy:—

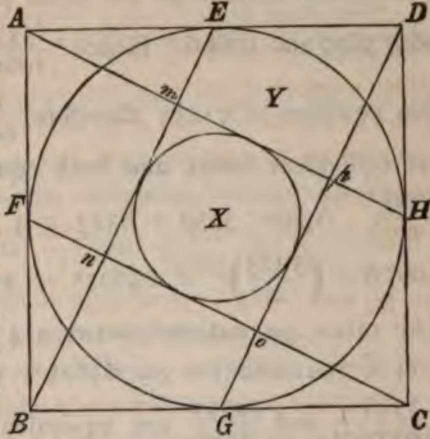
21st September, 1867.

To the Editor of the Athenæum.

Sir,

In your publication of to-day, I observe in notices to Correspondents, your acknowledgment of the receipt of my Letter of the 14th inst.

Now, Sir, I recognise, and admit, the distinction between you as the Editor of, and Professor de Morgan as the mathematical critic to, the leading Scientific Journal, and I know that you, Mr. Editor, are sufficient of a Mathematician to "*read and understand*" the facts that I am about to bring under your notice : and the question I shall put to you, arising out of these facts, you are competent to answer, without the assistance of either Professor de Morgan, or any other of our "*great mathematical authorities.*"



In this geometrical figure let A B C D represent a square. Bisect the sides at the points F, G, H, and E. Join E B, D G, A H, and F C. In the square *m n o p* inscribe the circle X, and in the square A B C D inscribe the circle Y. Then: The square *m n o p* is equal to one-fifth part of the square A B C D, and it follows of necessity, that the area of the circle X inscribed in the square *m n o p*, is equal to one-fifth part of the circle Y inscribed in the square A B C D.

Now, let the area of the circle X be represented by any finite arithmetical quantity, and be given to find two pairs of numbers, so that the smaller number in either pair shall be an aliquot part or submultiple of the larger number in the other pair, and the mean proportional between both pairs of numbers of the same arithmetical value.

Well, then, let the area of the circle X be represented by 666 the number of the Apocalyptic beast. Then: One pair of numbers will be 3330 and 832·5, and the other pair of numbers will be 2601·5625 and 1065·6, and $\sqrt{(3330 \times 832{\cdot}5)} = \sqrt{(2601{\cdot}5625 \times 1065{\cdot}6)}$, and this equation $= \sqrt{2771225} = 1665$; that is, the mean proportional between 3330 and 832·5, is represented by the same

arithmetical symbols as the mean proportional between 2601·5625 and 1065·6. Hence: $\frac{3330}{1065·6} = \frac{2601·5625}{832·5}$, and this equation $= 3·125$; therefore, $\frac{3330}{1065·6}$ and $\frac{2601·5625}{832·5}$ are equivalent ratios, and both equivalent to the ratio $\frac{3·125}{1}$. Again: $3330 \div 832·5 = 4$; and $2601·5625 \div 1065·6 = \left(\frac{3·125}{2}\right)^2 = 1·5625^2 = 2·44140625$. Hence: The mean proportional between 4 and $2·44140625 = \sqrt{(4 \times 2·44140625)} = \sqrt{9·765625} = 3·125$. But further: $\frac{3330}{2601·0625}$ and $\frac{1065·6}{832·5}$ are equivalent ratios, and both equivalent to the ratio $\frac{1·28}{1}$; and $1.28 : 1 :: 1 : ·78125$; therefore, $\frac{1·28}{1}$ and $\frac{1}{·78125}$ are equivalent ratios, and both express the ratio between the area of every square, and the area of its inscribed circle.

Again: Let the area of the circle X be represented by the digit 5, which stands in the centre of our system of arithmetical notation, and is the arithmetical mean between the extremes of 1 and 9, 2 and 8, 3 and 7, and 4 and 6. Then: One pair of numbers will be 25 and 6·25; and the other pair of numbers will be 19·53125 and 8; and, $\sqrt{(25 \times 6·25)} = \sqrt{(19·53125 \times 8)}$, and this equation $= \sqrt{156·25} = 12·5$. Hence: The mean proportional between 25 and 6·25 is represented by the same arithmeticial symbols as the mean proportional between 19·53128 and 8, and is equal to $4\ \pi$. Hence: $\frac{25}{8}$ and $\frac{19·53125}{6·25}$ are equivalent ratios, and both equivalent to the

ratio $\frac{3{\cdot}125}{1}$. Again: $25 \div 6{\cdot}25 = 4$; and $19{\cdot}53125 \div 8 = \left(\frac{3{\cdot}125}{2}\right)^2 = 1{\cdot}5625^2 = 2{\cdot}44140625$. Hence: The mean proportional between 4 and $2{\cdot}44140625 = \pi$, as in the previous example; and the remaining conclusion follows of necessity; that is, that $\frac{1{\cdot}28}{1}$ and $\frac{1}{{\cdot}78125}$ are equivalent ratios, and express the true ratio between the area of a square and the area of its inscribed circle.

Now, my good Sir, there is no mathematical magic in finding two pairs of numbers from a given area of the circle X, of which the mean proportional between the two pairs of numbers shall be of the same arithmetical value. All that is required is to know the *rule*, and I put the following plain question to you: Do you know the *rule?* If you know it, will you not, as an honest Scientific Journalist, let your readers have the benefit of your knowledge? If you know it not, pray have the candour to admit it! I know the *rule*, and have no wish to keep it a secret, and shall have much pleasure in revealing the *secret* to you, on receiving your assurance that you will give currency to it through the columns of the *Athenæum*.*

I am, Sir,

Yours respectfully,

J——S S——H.

* At the last meeting of the British Association, at Dundee, I offered to read a Paper in the Mathematical and Physical Section on mean-proportionals; but the Committee decided that such a subject was unsuited for the Section.

The fact is, the Committee knew me, and fancied that if they allowed me to read a Paper, "*π would be lugged in somehow*," and consequently declined to permit me to read my Paper.

These Letters, to Mr. Editor, were handed over to that *vulgar, contemptible, and untruthful twaddler*, Professor de Morgan, and drew from his pen the notice that disgraced the columns of the *Athenæum*, of the 28th September, 1867*." You must know, for I assure you of the fact, that de Morgan himself was the writer of the *Christmas Quadrature Carol* which appeared in the *Correspondent* of December 30, 1865.

*FROM THE "ATHENÆUM," *September 28th*, 1867.

"Bless the man!" said Miss Trotwood, of Micawber, "he would write letters by the ream if it were a capital offence." Mr. James Smith would do as much even were it a reasonable thing; so fond is he of writing. Fifteen quarto pages, mostly of geometry, in answer to our "complimentary" notice of his letter to the Duke of Buccleuch. Complimentary he calls it, because we have left his arguments unanswered; surely he does not mean that this is the first compliment we have paid him? And ridicule is not argument: we know that; if ridicule had been argument, we should never have ridiculed Mr. James Smith. He calls us to repentance and to $3\frac{1}{8}$; he says his letter is written to test our sincerity: how he does forget! he found out that we were imposters long ago. In one point he has hit us; we wrote *while* instead of *wile*: we remember how it was; we began to write *wheedle*, and changed it into *w(h)ile* in the act of writing. And so we are to *argue* with a man who produces a pentagon of which four sides are together *geometrically* larger than the fifth, but *mathematically* equal to it. Surely the *argument* of the Carol which appeared in the *Correspondent* is good enough. The first verse is as follows:—

A creed is a very fine thing—
Above all when there is'nt much of it;
There is no π but three and one-eighth,
And Mr. James Smith is its prophet.
  So here we go round, round, round,
  And there we go square, square, square,
  Five per cent. is a bob in the pound,
  And sixpence an omnibus fare.

3rd April, Afternoon.

I had written so far, when your favour of yesterday came to hand, enclosing Mr. R——'s Paper in reply to my Letter of the 26th March. He observes: "*As soon as* Mr. Smith *proves that π is determinate, and that $3\frac{1}{8}$ is the only number that fulfils certain conditions, or that $\frac{24}{25}$ circumference = perimeter of a regular inscribed hexagon; as soon as any of these things are proved by reasoning, not assumed or asserted, I shall then, if he desires, shew him how defectively he states such a problem as that he sends me for solution, and how much more simply the thing can be done, and how much construction he wastes.*" These are bold assertions. They are either true or false. If true, let Mr. R—— prove them. There can be no occasion to wait until I have fulfilled the conditions he imposes, even supposing me not to have fulfilled them already. Permit me to tell you, my dear Sir, that I doubt Mr. R——'s capacity to construct a regular octagon within a circle, from a given equilateral triangle, in any other way than that I have shewn him. If he can, let him prove it. If he prove it, he will at once shew that I am but a geometrical *quack*, and not a "*reasoning geometrical investigator.*" Let there be no evasion in this matter. Well, then, he imposes three conditions. Now, if Mr. R—— will read my

We have been told on good authority that the last line of the chorus was at first—

And J—— S—— is a d——y who's there.

Really Mr. J. S. deserves the restoration of the original reading, so painful as it is to us to make it known. *(Our Weekly Gossip.)*

Letter of the 30th March and this communication with care—unless, indeed, it be his misfortune to possess a mind utterly impervious to reason—he will find that I have favoured him with proofs of the first and second, and the third follows of necessity. But, I will give him a proof of the third, which I have never advanced before.

Well, then, on pages 4 and 5 of this communication, (see p. 90.) I have proved that the area of the circle X Y = 1·92 times the area of the circle Y, whatever be the value of π; and for all practical purposes we divide the circumference of a circle into 360 equal parts, called degrees. Now, $\frac{360}{3\cdot125}$ = 60 (1·92), and this equation = 115·2 = diameter of a circle of circumference 360; therefore, 3(115·2) = 345·6 = the perimeter of a regular inscribed hexagon to a circle of circumference 360. Now, I beg to refer Mr. R—— to my Letter to De Morgan, of October 31, 1864, or, to my Letter to you of the 3rd February, for a peculiar property of the circle, to which I directed your attention in that Letter; and then, if Mr. R—— be "*a reasoning geometrical investigator*," he will understand what follows. Now, divide the circumference of the circle into 180 equal parts. Then: $\frac{360}{180} = 2$; $\{2 - \frac{1}{25}(2)\} = 2 - \cdot08 = 1\cdot92$; $180\ (1\cdot92) = 345\cdot6$ = the perimeter of a regular inscribed hexagon to a circle of circumference 360; and, $\frac{345\cdot6}{6} = 57\cdot6$ = radius of a circle of circumference 360. Hence: We get the following equation: $\frac{3\cdot125}{3}(345\cdot6) = \frac{3\cdot125}{3}(180 \times 1\cdot92)$: and this equation

$= 360 =$ circumference of the circle. But, we get another equation: $\frac{3 \cdot 125}{3}\left(\frac{d}{2}\right) = \frac{3 \cdot 125}{3}\left(30(1 \cdot 92)\right)$, and this equation $= 60 = \frac{\text{circumference of circle}}{6}$; and it follows of necessity, that $\frac{3}{3 \cdot 125}$ $(60) = 57 \cdot 6 = \frac{\text{perimeter of hexagon}}{6}$ $=$ radius of the circle of circumference 360. But, we also get another equation: $\frac{\text{Perimeter of hexagon}}{6} =$ $\frac{4}{25}$ (circumference), that is, $\frac{345 \cdot 6}{6} = \frac{4 \times 360,}{25}$ and this equation $= 57 \cdot 6 =$ radius of the circle; and it follows of necessity, that the equation $\frac{4}{25}$ (circumference) $=$ $\frac{\frac{24}{25}\text{ (circumference)}}{6}$ — $= 57 \cdot 6 =$ radius of the circle.

Now, $3 \cdot 125\ (57 \cdot 6^2) = 3 \cdot 125 \times 3317 \cdot 76 = 10368 =$ area of a circle of circumference 360. This I have proved in previous Letters in several ways. If Mr. R—— dispute this, let him tell us what is the area of a circle of circumference 360? He will not *dare* to dispute that such a circle has an area, or say, that π has not an arithmetical value! Then, is it either disrespectful, unfair, or unreasonable, to ask Mr. R—— to tell us what is the area of a circle of circumference 360? I suspect, however, he will say,—if he say anything—"*I am not going to do anything of the sort, or of any sort;*" but I *dare* him to prove, that to ask this favour at his hands, is to "*divert attention from the point at issue between us.*"

Again: Referring to the enclosed diagram (*See Diagram VI.*) let AB the radius of the circle X$=57 \cdot 6$. Mr. R—— has admitted that $\pi r^2 =$ area in every circle. Well, then, $3 \cdot 125\ (AB)^2 = (AB^2 + BC^2 + AC^2)$, and this equation $= 10368$

= area of the circle X; therefore, 3 10368 = 31104 = area of the circle X Z. Proof: $\sqrt{}$(diameter^2 of circle X — radius^2 of circle X) = $\sqrt{(115.2^2 - 57.6^2)}$ = $\sqrt{(13271.04 - 3317.76)}$ = $\sqrt{9953.28}$ = radius of the circle X Z; and $\pi\ (r^2)$ = area in every circle; therefore, $\pi\ (\sqrt{9953.28}^2) = 3.125 \times 9953.28 = 31104$ = area of the circle X Z. Now, I have proved in a previous part of this communication, that 3 times the area of the circle X = area of the circle X Z, whatever be the value of π; and since no other value of π but $3\frac{1}{8}$ can fulfil these conditions, it follows, that 3·125 must be the true value of π; and I put the following question to Mr. R——: How in the name of common sense can π be otherwise than a *determinate* arithmetical quantity?

Now, my dear Sir, I may tell you, that Mr. R—— is nearly as disrespectful as Professor de Morgan. The only difference between them is this: The former makes me chargeable *(by implication)* with ignorance or insanity; the latter *(by implication)* makes me both ignorant and insane. But, never mind, I can stand to be *hit* hard, and afford to smile, and be generous. I am simply battling for truth, and as you said in your last, "*Magna est veritas et prevalebit, in the end;*" and so, as long as there is a shot left in the locker, "*never say die*" shall be my motto.

Mrs. Smith and I are glad to hear that the fine weather enables Mrs. S—— to get out for a walk, which will no doubt be of great service to her: and with our united kind regards to her, yourself, and daughter,

Believe me, my dear Sir,

Very faithfully yours,

J—— S——, Esq. JAMES SMITH.

DUMFRIESSHIRE,
2nd April, 1868.

My Dear Sir,

You are, indeed, untiring in your efforts to convince my relative, Mr. R——, who, however, evidently chimes in with Professor de Morgan, in his view of your effort at squaring the circle. *Petitio Principii* he, unhesitatingly, charges you with, and, if you see no way of rebutting this charge to his satisfaction, I fear the correspondence will fall to be closed.

Admiring your pluck, I enclose his production of *All fools' day!* Yours of 30th ult., he would get to-day.

I remain, my dear Sir,
Yours faithfully,
J—— S——.

JAMES SMITH, Esq.,
Barkeley House.

Mr. R——'s PAPER, *April 1st*, 1868.

I have only now found time to notice Mr. Smith's Paper. It is nothing to the point. If I refuse to do a task set me by Mr. Smith, the reason of my refusal is neither inability to do a thing so simple, nor is it a want of candour. Free remark of the kind Mr. Smith indulges in about those who see nothing solid in his quadrature, who see that it is a mere theory or guess that cannot be proved, is nothing to the point. This way of discussing a mathematical point is inadmissible.

As soon as Mr. Smith *proves* that π is determinate, and that $3\frac{1}{8}$ is the only number that fulfils certain conditions, or that $\frac{24}{25}$ (circumference) = perimeter of a regular hexagon, as soon as

any of these things is proved by reasoning—not assumed or asserted—I shall then, if he desires, show him how defectively he states such a problem as that he sends me for solution; and how much more simply the thing may be done, and how much construction he wastes. But I am not going to do anything of the sort, or of any sort that will divert attention from the point at issue between us.

I may repeat what I said once before, that practical geometry can prove nothing. It can find its place, and Mr. Smith any amount of scope for ingenious constructions, when he has established the true value of π. In practical geometry we *apply* truths previously established. For instance, we find the centre of a circle by taking any two points in the circumference, joining the points, bisecting the line of junction, from the point of bisection drawing a line at right angles to meet the circumference at opposite sides, and then bisecting that line; but how many things that are previously established and known to be true by pure reasoning, are involved in this simple process. It would be a very easy thing to construct a square = to a circle, if only we knew the ratio of the diameter to the circumference; but this ratio must be established before Mr. Smith's ingenuity in construction can be anything higher than ingenious play. The ingenuity of construction is valuable; but I care not how rough the figures, if we have the pure demonstration, which can be evaded only by ignorance or insanity.

Mr. R——'s Paper, *April 2nd*, 1868.

I regret that I have allowed a little asperity of expression to intrude into some of my notes on Mr. Smith's Papers. If I knew him as well as he knows the mathematicians, perhaps I would bear with more equanimity what seems unreasonable in his reasonings and in his "moves." I may be allowed to say, however, that with me it is no game of chess, or game at

all, but simply analysis of what professes to be proof of the true ratio of diameter to circumference.

With respect to the problem set me, I am rather amused that so much, that anything, is made of it. The solution given by Mr. Smith is not the only solution that would satisfy the thing he proposes; because the terms of that proposition are quite indefinite.* I shall, to satisfy Mr. Smith, criticise this proposition of his: but in the meantime, let us see whether it supplies anything new or more satisfactory than former efforts, of proof that $\pi = 3\frac{1}{8}$.

Now, that $x : z :: 1 : 3\frac{1}{8}$ is a result so simple, that you have only to state the proportion to see it: circle area being as the squares of radii, $x : z :: 16 : 50 :: 1 : 3\frac{1}{8}$. π does not appear, need not appear. No other proof than this simple statement of the ratios, is needed to show that $3\frac{1}{8}(x) = z$, when the radius of $x = 4$, and the radius of $z = (5\sqrt{2}.)$ But how does this upset the *fact* that π is indeterminate? This shows nothing *whatever* as to the value of π. $\begin{pmatrix} 3\frac{1}{8}\,\pi \times (4^2) = \pi \times (5\sqrt{2})^2 \\ 3\frac{1}{8} \times 16 \cdot \pi = 50 \cdot \pi \end{pmatrix}$; this is the sum of it: that $50\,\pi = 50\,\pi$, that $50 = 50$, and that $\pi = \pi$. But for anything that *this ratio* or *this equation* shows, π may be determinate or indeterminate: *may be* any number between one and a million.† How can Mr. Smith say that this fact, $3\frac{1}{8}(x) = z$, even if it stood alone, without confirmation of any kind, upsets the "*theory*" that π is indeterminate? This is perfect nonsense.

But Mr. Smith goes on to prove that $\pi = 3\frac{1}{8}$ in the following way: $3\frac{1}{8}(OB^2 + BC^2 + OC^2) = 12\frac{1}{2}\left(\frac{OD}{2}\right)^2$; *i.e., he says,* $3\frac{1}{8}$ *times the squares of the sides of* O B C = *area of the circle z.* This is the old fallacy and assumption over again. Mr. Smith has *said* this over and over, in the Buccleuch Letter, and in other Papers. I once more dispute this equation; and request Mr.

* This proves that Mr. R—— was in possession of my solution of the problem before he favoured me with his.

† Mr. R—— should have proved how π *may be* any number between one and a million, and be indeterminate.

Smith to look candidly at the fact, that it is really ***taking for granted*** the *very thing* in dispute. This I boldly assert. Nobody questions that $12\frac{1}{2}\left(\frac{OD^2}{4}\right) = 3\frac{1}{8}(OB^2 + BC^2 + OC^2)$. But what is questioned is that they are also = area of circle *z*.

They *would be so* IF $3\frac{1}{8} = \pi$; but that does not *prove* $\pi = 3\frac{1}{8}$. I am now perfectly sure that this cannot be proved, though I do not expect Mr. Smith to yield the point now that he has disputed these 7 or 8 years with "*savans.*" Has he convinced any one? Has any of all with whom he has been in correspondence, been made to see that $\pi = 3\frac{1}{8}$? If so, where is the testimony of those who believe? Some of them might succeed with a stubborn man like me, better than Mr. Smith himself. Such a person might show how he was convinced!

To me it is perfectly inexplicable how Mr. Smith should persist so confidently in believing without a shade of proof. This is not an intelligent faith. $3\frac{1}{8}$ is a *neat* result; but $3\frac{1}{5}$ is as *neat*. My parody of the Buccleuch demonstration should have shown Mr. Smith that his proof is no proof; for it exhausted all the steps in his that can be called essential. I trust that Mr. Smith will not take my freedom amiss.

A few words now about the proposition: "From an equilateral triangle, construct a regular octagon within a circle." I give 3 figures. In No. 1. a circle circumscribes an equilateral: in No. 2. the apex of the equilateral is centre. I need not show how in Figs. 1 and 2 an octagon may be inscribed from the triangle. Either case would meet Mr. Smith's enunciation; an octagon would be inscribed in a circle from an equilateral triangle. Look at No. 3. O A B is equilateral, O C is drawn perpendicular to O B; B C is joined, and bisected in D. Join O D, and produce O D and O B indefinitely; and with O as centre and any radii, describe circles. Need I show that the chords of the arcs of these circles, intercepted by the lines O B, O D produced, will all be sides of an octagon in each circle respectively? And need I show how the figure is to be completed?

Now, here is any number of solutions of Mr. Smith's problem, so indefinite is the enunciation of it. The problem should be more definitely worded. He should have said: Inscribe an octagon in a circle whose radius shall bear to the radius of another circle, or to the side of an equilateral triangle, a given proportion; here it is: as $4 : 5(\sqrt{2})$ or, as $1 : \frac{5}{2(\sqrt{2})}$. But what has the equilateral to do with it? In Mr. Smith's, and in No. 3 here, no use is made of the equilateral. It is really a nonsense kind of thing after all. Mr. Smith makes no use of the equilateral, and none of the octagon, in his *demonstration* or process. If he prove that the squares $OB^2, + BC^2 + OC^2 = x$, all right. That would establish his ratio. But, I must boldly assert that this cannot be proved, because it is not true; and the figure constructed is of no use whatever. It is merely like shuffling the cards in a card trick. It is the same old assumption and assertion slightly altered in the way of putting it.

MR. SMITH *to* MR. J—— S——.

BARKELEY HOUSE, SEAFORTH,
6th April, 1868.

MY DEAR SIR,

I am in receipt of your favour of the 4th, enclosing Mr. R——'s remarks on my Letter of the 30th March, for which accept my best thanks. I am bound to admit that Mr. R—— has "*planted*" a *hard hit* this time. He shows us how to construct a regular octagon within a circle, and divide the angle of a quadrant into two angles of 45° each. This construction was familiar enough to me at one time, but I must honestly confess, it was not present to my mind when I gave Mr. R—— the problem for solution; nevertheless, I can assure him, the problem

was not given as a "*joke.*" I was clearly too intent upon my own game, and the moves to follow in it, and so, lost sight of Mr. R——'s counter-move. But, never mind; after all, the cause of scientific truth may be more advanced than retarded by my lapsus. I might, and as it now appears, should have "*stopped*" Mr. R——'s blow, by putting the problem in the following way:—

PROBLEM.

Construct a regular octagon within a circle from a given equilateral triangle, and so, that we may then divide the angle of a quadrant of the circle into 3 angles, of which an angle of two similar triangles in the figure shall be one of these angles.

Now, although my method of constructing the octagon, as shown in the diagram enclosed in my Letters of the 26th and 30th March, differs from Mr. R——'s mode of construction, he does not say that I do not get a regular octagon within a circle from a given equilateral triangle, but complains that I make no use of the equilateral. In my next I will show Mr. R—— the use I make of the equilateral; in the meantime, I enclose a diagram which is a fac-simile of those already referred to, with the following additions. Produce C B to meet the circumference of the circle Y, at the point P, and join O P; draw the radius O T at right angles to the radius O E. *(See Diagram VII.)*

Then: T O E is a quadrant of the circle Z, and is divided into three angles, T O P, P O F, and F O E. Now, will Mr. R—— be good enough to tell us the values of these three angles expressed in degrees, and the

arithmetical values of their natural sines? Some remarkable results shall follow, by certain additions to the diagram. With best wishes,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. J—— S—— *to* Mr. JAMES SMITH.

DUMFRIESSHIRE, *7th April*, 1868.

My Dear Sir,

Having forwarded your voluminous despatch, on receipt and perusal, I, this morning, forwarded yours of yesterday to my relative, Mr. R——. He will expect your promised successor to follow. Your candid confession should please him.

Meantime, my brother having since sent me the enclosed somewhat homely budget, I send it, with the part of his accompanying short note relating thereto.

The *Athenæums* did not at all surprize me, after having read in the same periodical, Professor de Morgan's article, and the Review of a book of another relative, which, so far as I know, was not unfavourably reviewed from any other quarter.

You, in your search after truth, are however, well prepared for such attacks, and can afford to laugh at them.

Believe me, my dear Sir,

Faithfully yours,

J—— S——.

JAMES SMITH, ESQ.,
Barkeley House.

Mr. JAMES SMITH *to* Mr. J—— S——.

BARKELEY HOUSE, SEAFORTH,
11*th April*, 1868.

MY DEAR SIR,

Mr. R—— commences his last Paper by observing :—"*I regret that I have allowed a little asperity of expression to intrude into some of my notes on Mr. Smith's Papers :*" and concludes by saying : "*I trust that Mr. Smith will not take my freedom amiss.*" I can assure Mr. R—— I will not take "*amiss*" anything he may say, and I will not say anything *intentionally* to give offence. I must crave the same indulgence at his hands. We are fighting a *great battle*, and we must both be permitted to "*plant*" a "*hard hit*" when we get the opportunity.

Mr. R—— observes :—"*Now, that* $X : Z :: 1 : 3\frac{1}{8}$ *is a result so simple that you have only to state the proportion to see it. Circle area being as the squares of radii,* $X : Z :: 16 : 50 :: 1 : 3\frac{1}{8}$. π *does not appear, need not appear.*" Does not $3\frac{1}{8}$ appear? And since $3\frac{1}{8}\ (X) = Z$, by whatever finite value of π we get the values of X and Z, how can the value of π be indeterminate? Surely Mr. R—— should have exercised a little reflection, before he said of my conclusion, "*this is perfect nonsense.*"

Now, my dear Sir, mark the *dilemma* into which Mr. R—— has brought himself! He will not venture to dispute, that the area of a circumscribing square to any circle is found, by dividing the area of the circle by $\frac{\pi}{4}$, whatever be the value of π. Now, if a denote the area of a circle, and b the area of a circumscribing square ; then, $\frac{a}{\frac{1}{4}\pi}$ =

b, whatever be the value of π ; and it follows of necessity, that if π be indeterminate, *b* must also be indeterminate. The conclusion to which Mr. R——'s argument and reasoning would lead us would be this : The diameter of an inscribed circle to any square cannot be arithmetically expressible, either by a finite number or the square root of a finite number. Mr. R—— should have thought of this, before he ventured to say of my conclusion: "*This is perfect nonsense.*"

Well, then, Mr. R—— admits the following fact : "*No other proof but the simple statement of the ratios, is needed to shew that* $3\frac{1}{8}$ X = Z, *when the radius of* X = 4, *and the radius of* Z = 5 ($\sqrt{2}$); " and he then puts the question : "*But how does this upset the fact that* π *is indeterminate?*" Surely Mr. R—— will now see, that in *assuming* it to be a *fact* that π is indeterminate, he might as well assume that the area of a square is indeterminate! Could absurdity go further? Well, then, it is not necessary to analize Mr. R——'s last Paper with greater severity ; but, I may assure him, it has greatly amused me.

I must now proceed to show Mr. R—— the use I make of the "*equilateral.*" The enclosed diagram is a fac-simile of that contained in my Letter of yesterday, with the following additions. From the angle O of the equilateral triangle O A B, let fall a perpendicular bisecting its opposite side A B at H. From the angle H, draw a straight line parallel to O A, to meet and bisect O B at K. Join P F. With O as centre and O F as interval, describe the circle M. With O as centre and O H as interval, describe the circle N ; and with O as centre and O K as interval, describe the circle P. *(See Diagram VIII.)*

Now, let O K = 2. Then : O H = $\sqrt{12}$; O B = 4;

$O C = 5$; $O F = 6{\cdot}25$; and $O D = \sqrt{50}$. Of these facts Mr. R—— may readily convince himself; and he *knows* that $O D = 5(\sqrt{2})$, that is, that $O D = 5$ times the diagonal of a square of which the side is unity $= 1$.

Then:

$\pi (O K^2) = 3{\cdot}125 \times 4 = 12{\cdot}5 =$ area of the circle P.

$\pi (O H^2) = 3{\cdot}125 \times 12 = 37{\cdot}5 =$ area of the circle N.

$\pi (O B^2) = 3{\cdot}125 \times 16 = 50 =$ area of the circle X.

$\pi (O C^2) = 3{\cdot}125 \times 25 = 78{\cdot}125 =$ area of the circle Y.

$\pi (O F^2) = 3{\cdot}125 \times 39{\cdot}0625 = 122{\cdot}0703125 =$ area of the circle M.

$\pi (O D^2) = 3{\cdot}125 \times 50 = 156{\cdot}25 =$ area of the circle Y.

Hence: If the letters which represent the circles, also denote their superficial areas: $3 (P) = N$; $4 (P) = X$; $6{\cdot}25 (P) = Y$; $3{\cdot}125^2 (P) = M$; and $12{\cdot}5 (P) = Z$. (That these are the true ratios—or relative values—of area to area in the circles, may be demonstrated by means of any finite hypothetical value of π, intermediate between 3 and 4) Hence: $3\frac{1}{8} (X) = P^2$, and this equation $=$ area of the circle Z. How, I would ask Mr. R——, could this be possible on any value of π, if π were only arithmetically expressible by an infinite series?

We now get the following equations: $4 \pi (P) = \pi (O D^2)$, and when the radius of the circle $P = 2 = 156{\cdot}25$; therefore, the equation $4 \pi (P) = \pi (O D^2) =$ area of the circle Z. $4^2 (P) = 4 (O D^2)$, and when the radius of the circle $P = 2 = 200$; therefore, the equation $4^2 (P) = 4 (O D^2) =$ area of a circumscribing square, to the circle Z.

Now, to controvert the *theory* that 8 circumferences $= 25$ diameters in every circle, which makes $\frac{25}{8} = 3{\cdot}125$

the value of π ; Mr. R—— must find some other value of π by which he can get the equations, $4\,\pi\,(P) = \pi\,(O\,D^2)$, and $4^2\,(P) = 4\,(O\,D^2)$. Let him try, and where will he be? Certainly not "*in π glory!*" Mr. R—— must not take this "*amiss.*" I got the idea from Professor de Morgan, and can assure Mr. R—— that I mean no offence by *making use* of it.

Again: The area of the circle P = half the area of the square O C D G, and it follows of necessity, that the area of the circle P = area of a square on the half of O D the diagonal of the square O C D G. But, O D the diagonal of the square O C D G $= 5\,(\sqrt{2}) = \sqrt{50}$, when O K the radius of the circle P = 2; therefore, $\frac{1}{2}\,(\sqrt{50^2}) = \sqrt{12\cdot5^2} = 12\cdot5$ = area of a square on the semi-radius of the circle Z; and $4\,\pi\,(s\,r^2)$ = area in every circle; therefore, $4\,\pi\,(12\cdot5) = 12\cdot5 \times 12\cdot5 = 12\cdot5^2 =$ area of the circle Z; and since the property of one circle is the property of all circles, it follows of necessity, that $12\frac{1}{2}$ times the area of a square on the semi-radius = area in every circle.

Again: When O K the radius of the circle P = 2, O C a side of the square O C D G = 5, by construction; therefore, $4\,(O\,C) = 20$ = perimeter of the square O C D G; therefore, $\frac{4\,(O\,C)}{5} = \frac{20}{5} = 4$ = diameter of the circle P. "*π does not appear, need not appear.*" Can Mr. R—— prove that $12\frac{1}{2}$ times the area of the circle P is equal to $4\,\pi\,\left(\frac{O\,D}{2}\right)^2$ whatever be the value of π? Certainly not! Well, then, if Mr. R—— will persist in asserting that π is indeterminate, let him at any rate furnish some *rational* proof of it. Where

is such proof to be found? His saying it is so, cannot make it so.

Well, then, in my Letter of the 30th March, I made no use of the "*equilateral;*" but, I have now made some use of it; and I suspect that Mr. R—— will think twice before he reiterates the *assertion* that the use I have made of it "*is really a nonsense kind of thing after all.*" Will Mr. R—— any longer hesitate to admit, that $(OB^2 + BC^2 + OC^2) = X$? He does admit that if this equation be "*all right,*" my "*ratio is established.*" If this equation be not "*all right,*" let Mr. R—— prove it by some *rational* demonstration! Will he attempt to do so? We shall see!!

I shall now proceed to make use of the "*equilateral,*" for the purpose of shewing Mr. R—— that $\frac{24}{25}$ (circumference) = perimeter of a regular inscribed hexagon in every circle.

How many times have I stated the *fact*, that $\frac{\pi}{4} = 12{\cdot}5\left(\frac{\pi}{50}\right)$, and that this equation = area of a circle of diameter 1, whatever be the value of π. Where is the Mathematician who *dare* dispute this fact? Where is the Mathematician who *dare* tell me that I cannot be permitted to adopt this fact as a *datum*? Will Mr. R—— venture to tell me that such a *datum* is inadmissable, as regards the question at issue? Do not Mathematicians *assume* as a *datum* that the value of π can be ascertained by means of multilateral polygons inscribed and circumscribed to a circle of radius 1? I dispute their *assumption*. This *datum* is not self-evident, neither is it *rational*, and I ask them for a *rational* proof of it. Can they furnish it? I trow not! The *datum* $\frac{\pi}{4} = 12{\cdot}5\left(\frac{\pi}{50}\right)$ is *indisputable* and *rational*. Where is the

Mathematician who *dare* dispute this? Well, then, let us take this *datum* for a starting point.

Now, let r denote the radius, and c the circumference of a circle of diameter unity $= 1$. Then: $\frac{25}{24}(r) \times$ twice the perimeter of a regular inscribed hexagon $= r \times 2\pi$, that is, $\frac{25}{24}(r) \times 6 = r \times 2\pi$. Hence: $\frac{24}{25}(c) = 6(r)$; and this equation $=$ the perimeter of a regular inscribed hexagon to a circle of diameter unity; and, $\frac{4}{25}(c) =$ radius of a circle of diameter unity; therefore, $\frac{8}{25}(c) =$ diameter of the circle, and makes $\frac{25}{8} = 3{\cdot}125$, the value of π. This harmonizes with our *datum*, that is, $12{\cdot}5\left(\frac{\pi}{50}\right) = \frac{\pi}{4}$ or, $12{\cdot}5\,({\cdot}0625) = \frac{3{\cdot}125}{4}$, and this equation $= {\cdot}78125 =$ area of a circle of diameter unity.

Proof: Circumference $\times$ semi-radius $=$ area in every circle; therefore, $C \times \frac{r}{2}$, that is, $3{\cdot}125 \times {\cdot}25 = {\cdot}78125$ $=$ area of a circle of diameter unity. Take another proof. When the diameter of the circle $X = 1$, OB, the radius $= {\cdot}5$, and $BC = \frac{3}{4}(OB) = {\cdot}375$, by construction, and B is a right angle; therefore, $OB^2 + BC^2 = {\cdot}5^2 + {\cdot}375^2 = {\cdot}25 + {\cdot}140625 = {\cdot}390625 = OC^2$; therefore, $(OB^2 + BC^2 + OC^2) = {\cdot}25 + {\cdot}140625 + {\cdot}390625) = {\cdot}78125 =$ area of the circle X, when the diameter $= 1$. Hence: $(OB + BC^2 + OC^2) = 3\frac{1}{8}(OB^2)$, and this equation $= \frac{\pi}{4} = 12{\cdot}5\left(\frac{\pi}{50}\right) = {\cdot}78125 =$ area of the circle, and harmonizes with our *datum*.

Can Mr. R—— find any other value of π, but that which makes 8 circumferences $=$ 25 diameters in every circle, by which he can produce these results? I trow not! Then, will he reiterate his assertion that π is inde-

terminate? Will he reiterate his assertion that $\frac{24}{25}$ (C) is not equal to the perimeter of a regular inscribed hexagon in every circle? Will he reiterate his assertion that the equation $(O B^2 + B C^2 + O C^2) = 3\frac{1}{8} (O B^2)$, which = area of circle X, "*cannot be proved, because it is not true*"? Will he again tell me, that "*the figure constructed is of no use whatever*"? Will he again tell me that the use I make of it "*is merely like shuffling the cards in a card trick*"?

I must now refer you to my Letter of the 17th December, 1867—the first I ever wrote you on the subject of the Quadrature of the Circle, and was in reply to a Letter from you, enclosing diagrams from your brother, with the orthodox calculations for finding the value of π. This Letter commenced with the following paragraphs:—

"*I am in receipt of your esteemed favour of the* 13*th inst., and beg to thank you for the interest you have taken in my labours on the Quadrature and Rectification of the Circle.*

"*With your brother's calculations, enclosed in your Letter, I was quite familiar. The error involved in them is not one of calculation, but of principle. The* 47*th proposition of the first book of Euclid treats of a rectilinear figure, and is therefore inapplicable, directly, to measure a curvilinear figure; but* (indirectly) *it can be made available, in many ways, to prove the ratio of diameter to circumference in a circle.*"

Now, since the perimeter of a regular inscribed dodecagon to a circle—whether of radius 1 or diameter 1—is incommensurable, it follows, that the perimeters of all inscribed polygons, of a greater number of sides than 12,

must be incommensurable. Hence: If there were not a fallacy in the methods adopted by Mathematicians, for finding the value of π, it follows, that π would necessarily be an *indeterminate* quantity. Now, Mr. R—— boldly avows his faith in the orthodox methods of searching for π. I maintain—on the contrary—that all these methods are based on an *assumption*, which never has, and never can be proved; and so, there is a wide gulf between us on the very threshold of our enquiry, and if this difficulty cannot be got over, you cannot fail to see that further controversy is useless. Well, then, I will make another effort to get over this difficulty, and bring us to an admitted starting point.

Now, my dear Sir, I maintain that before attempting to find what the arithmetical value of π is, we must first dispose of a preliminary question, viz., is π a determinate or an indeterminate arithmetical quantity? You will at once perceive, that to *assume* π to be either determinate or indeterminate without proof, would be absurd; and the question arises:—Is it *demonstrable* that the true arithmetical value of π *must* be a finite quantity? To this question Mr. R—— answers, No! I answer, Yes! It has ever appeared to the Mathematician *absurd* and *irrational*, to suppose that π must be indeterminate: hence, the number of labourers in the field of circle-squaring in every age. Well, then, that the arithmetical value of π is *determinate* may be proved in many ways, but I know of no proof more likely to overcome the prejudices of a Mathematician, and convince an opponent, than one I can furnish by means of the enclosed diagram.

Well, then, O F is the radius of the circle M. Now, let m denote the area of this circle, and let n denote the area of a circumscribing square.

Then:

$$4\pi\left(\frac{O\,F}{2}\right)^2 = m\,;\ \text{and, } 16\left(\frac{O\,F}{2}\right)^2 = n.$$

$$\therefore n \times \tfrac{1}{4}\pi = m\,;\ \text{and, } \frac{m}{\frac{1}{4}\pi} = n.$$

But, when O B the radius of the circle X = 4, then, O F the radius of the circle M = 6·25; therefore, 4 (O F²) = 4 (6·25²) = 4 × 39·0625 = 156·25 = n; therefore, the value of n is a known and indisputable *determinate* quantity. Hence: It may be demonstrated that we can get the equation $\frac{m}{\frac{1}{4}\pi} = n$, by any finite hypothetical value of π, intermediate between 3 and 4.

For example: By hypothesis let π = 3·1416, and O B the radius of the circle X = 4, which makes O F the radius of the circle M = 6·25. Then: (πr^2) = area in every circle; therefore, π (O F²) = 3·1416 (6·25²) = 3·1416 × 39·0625 = 122·71875 = m; therefore, $\frac{m}{\frac{1}{4}\pi} = n$. Now, $\frac{122{\cdot}71875}{{\cdot}7854}$ = 156·25; and this equation = 4(O F²) = n. But, any other hypothetical value of π will produce the same result; and how then, in the name of common sense, can π be indeterminate, whatever be its true arithmetical value? This "*upsets*" the orthodox coach!!

Again: O F = 6·25, when O B the radius of the circle X = 4, and 2 π (radius) = circumference in every circle. Well, then, 2 π (O F) = O F² = 6·25² = 39·0625; therefore, 8 (O F²) = 25 (2 × O F), that is, 8 × 39·0625 = 25 × 12·5, and this equation = $100\left(\frac{O\,F}{2}\right)$ = 312·5; therefore, $\frac{312{\cdot}5}{100}$ = 3·125 is the true value of π, and makes 8

circumferences = 25 diameters in every circle. Hence: 39·0625 is the circumference of the circle M, when O B the radius of the circle X = 4, which makes O F = 6·25.

Well, then, take the formula:

$$4\,\pi\left(\frac{O\,F}{2}\right)^2 = m;\ \text{and, } 16\left(\frac{O\,F}{2}\right)^2 = n.$$

$$\therefore\ n \times \tfrac{1}{4}\pi = m;\ \text{and, } \frac{m}{\frac{1}{4}\pi} = n,$$

when m denotes the area of the circle M, and n denotes the area of a circumscribing square.

I shall now direct your attention to certain results, to which every other but the true value of π is utterly incompetent.

Let the diameter of the circle X be represented by unity, which makes O B the radius of the circle $= \frac{1}{2} = \cdot 5$, and makes O F the radius of the circle M $= \cdot 78125 = \frac{\pi}{4}$. Then: $4\,\pi\,(8\,r^2)$ = area in every circle; therefore, $4\pi\left(\frac{O\,F}{2}\right)^2 = 12\cdot 5\left(\frac{\cdot 78125}{2}\right)^2 = 12\cdot 5\,(\cdot 390625^2) = 12\cdot 5 \times$ $\cdot 152587890625 = 1\cdot 90734863281 25 = m$; therefore, $\frac{m}{\frac{1}{4}\pi}$ $= n$, that is, $\frac{1\cdot 9073486328125}{\frac{1}{4}\pi} = 2\cdot 44140625 = n$. But, $4\,(O\,F^2) = 4\,(\cdot 78125^2) = 4 \times \cdot 6103515625 = 2\cdot 44140625$ $= n$; therefore, $n = \left(\frac{\pi}{2}\right)^2$, that is, $n = \left(\frac{3\cdot 125}{2}\right)^2 = 1\cdot 5625^2 =$ $2\cdot 44140625$, and no other but the true value of π can produce this result.

Again: When the diameter of the circle X is represented by unity, O B the radius = ·5; and $2\,\pi\,(r)$ = circumference in every circle; therefore, $2\,\pi$ (O B) = $6\cdot 25 \times .5 = 3\cdot 125$ = circumference = π; therefore,

$8 (3 \cdot 125) = 50 (\cdot 5) = 25$, and makes 8 circumferences of the circle = 25 diameters, and no other but the true value of π can produce this result.

Now, you will observe that N T and L E diameters of the circle Z, cut the circumference of the circle M at the points *m*, *n*, *o* and *p*. On O *m*, O *n*, O *o* and O *p*, describe squares. In this way we get a circumscribed square to the circle M; and by joining *m n*, *n o*, *o p* and *p m*, we get an inscribed square to the circle M. The circumscribed square to the circle M will stand on the circle Z, and be exactly equal to it in superficial area.

First proof: When O B the radius of the circle X = 4, O D the radius of the circle $Z = 5(\sqrt{2}) = \sqrt{50}$; and O F the radius of the circle $M = 6 \cdot 25$; therefore, $\pi (O D^2) = 3 \cdot 125 \times 50 = 156 \cdot 25$ = area of the circle Z. But, $4 (O F^2) = 4 (6 \cdot 25^2) = 4 \times 39 \cdot 0625 = 156 \cdot 25$ = area of a circumscribing square to the circle M; therefore, the area of the circle Z, and the area of a circumscribing square to the circle M, are exactly equal.

Second proof: Let O C the radius of the circle Y be represented by any arithmetical quantity, say 10. Then: $C F = \frac{3}{4} (O C)$, and, $C D = O C$, by construction; therefore, $O C = 10$, and, $\frac{3}{4} (O C) = \frac{3}{4} (10) = 7 \cdot 5 = C F$; therefore, $O C^2 + C F^2 = 10^2 + 7 \cdot 5^2 = 100 + 56 \cdot 25 = 156 \cdot 25 = O F^2$; therefore, $\sqrt{156 \cdot 25} = 12 \cdot 5 = O F$ the radius of the circle M: and, $O C^2 + C D^2 = 10^2 + 10^2 = 100 + 100 = 200$; therefore, $\sqrt{200} = O D$ the radius of the circle Z. But, $4 (O F^2) = 4 (12 \cdot 5^2) = 4 \times 156 \cdot 25 = 625$ = area of a circumscribing square to the circle M, and is equal to $\pi (O D^2)$, that is, $4 (O F^2) = \pi (O D^2)$ or, $4 (12 \cdot 5^2) = 3\frac{1}{8} (\sqrt{200}^2)$; therefore, the equation $4 (O F^2) = \pi (O D^2) =$

area of a circumscribing square to the circle M. But, π $(O\,C^2) = 3{\cdot}125\ (10^2) = 3{\cdot}125 \times 100 = 312{\cdot}5 =$ area of the circle Y, and twice the area of the circle Y $= 2\,(312{\cdot}5)$ $= 625 =$ area of the circle Z ; therefore, $2\,\{\pi\,(O\,C^2)\} =$ $4\,(O\,F^2)$; that is, $2\,(3{\cdot}125 \times 10^2) = 4\,(12{\cdot}5^2)$, and proves that the area of the circle Z and the area of a circumscribing square to the circle M, are exactly equal.

Third proof: When the area of a circumscribing square to the circle M $= 625$, then, $\frac{625}{2} = 312{\cdot}5 =$ area of an inscribed square to the circle. Then: $\{(312{\cdot}5 + \frac{1}{4}\ 312{\cdot}5) + \frac{1}{4}\,(312{\cdot}5 + \frac{1}{4}\ 3{\cdot}125)\} = \{390{\cdot}625 + 97{\cdot}65625\}$ $= 488{\cdot}28125 =$ area of the circle M ; and, $488{\cdot}28125 \times$ $1{\cdot}28 = 625 =$ area of the circumscribing square to the circle M. "*π does not appear, need not appear:*" but he "*lies lurking in his den*"—an expression of my correspondent, Mr. Gibbons—and may be induced to shew himself, when we know how to go to work for the purpose. Well, then, $\frac{\pi}{4} : 1 :: 1 : 1{\cdot}28$; that is, $\frac{3{\cdot}125}{4} : 1 :: 1 : 1{\cdot}28$; therefore, $\frac{1}{{\cdot}78125}$ and $\frac{1{\cdot}28}{1}$ are equivalent ratios, and it follows of necessity, that the product of any arithmetical quantity multiplied by $1{\cdot}28$, is equal to the quotient of the same quantity divided by $\cdot78125$, and conversely. Hence: When the area of a circumscribing square to a circle $= 625$, then, $\frac{625}{1{\cdot}28} = 488{\cdot}28125 =$ area of the circle. But, when the area of a square $= 625$, the diameter of an inscribed circle $= \sqrt{625}$; therefore, $\frac{1}{2}\,(\sqrt{625}) = \sqrt{156{\cdot}25}$ $=$ radius ; and $\pi\,r^2 =$ area in every circle ; therefore, $\pi\,r^2 = 3{\cdot}125 \times 156{\cdot}25$ $488{\cdot}28125 =$ area of the circle.

But, $\frac{\text{area}}{\frac{1}{4}\pi}$ = area of a circumscribing square in every circle; therefore, $\frac{488 \cdot 28125}{\frac{1}{4}\pi} = \frac{488 \cdot 28125}{\cdot 78125} = 625$ = area of a circumscribing square. But, I have proved that the area of the circle Z = 625 when the radius = $\sqrt{156 \cdot 25}$; therefore, the area of the circle Z and the area of a circumscribing square to the circle M, are exactly equal.

Fourth proof: Let the area of the circle Z be represented by any arithmetical quantity, say 600. Then: Since πr^2 = area in every circle, it follows of necessity, that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle, whatever be the value of π. Mr R—— will not venture to dispute that the area of the circle Z may be represented by 600. Now, let m denote the area of a circumscribing square to the circle M, and = 600; that is, equal area of the circle Z. Let n denote the area of the circle M, and let p denote the area of an inscribed square to the circle M. Then: The diameter of the circle M = $\sqrt{600}$, and the radius = $\frac{1}{2}(\sqrt{600}) = \sqrt{150}$. Now, $\frac{600}{1 \cdot 28} = 468 \cdot 75 = n$; and, $\{(n - \frac{1}{5}n) - \frac{1}{8}(n - \frac{1}{5}n)\} = 375 - 75 = 300 = p$; therefore, $\sqrt{2p}$ = diameter of the circle = $\sqrt{600}$. But, $\sqrt{\frac{n}{\pi}} = \sqrt{\frac{468 \cdot 75}{3 \cdot 125}} = \sqrt{150}$ = radius of the circle, and $4(r^2) = 4 \times 150 = 600 = m$. But, $\sqrt{\frac{600}{\pi}} = \sqrt{\frac{600}{3 \cdot 125}} = \sqrt{192}$ = radius of the circle Z; and, $\frac{3}{4}(\sqrt{192}) = \sqrt{108}$. Now, let A B C denote a triangle, of which the sides AB and BC contain a right angle, and AB = $\sqrt{192}$ and BC = $\sqrt{108}$; then, AC, the hypothenuse, = $\sqrt{300}$; therefore,

$(AB^2 + BC^2 + AC^2) = 3\frac{1}{3}\ (AB^2)$, and, this equation = area of the circle Z, when the radius = $\sqrt{192}$. But, when OB the radius of the circle X = 4, the area of the circle M = 122·0703125, and the area of the circle Z = 156·25 ; therefore, 122·0703125 : 156·25 : : 468·75 : 600, and proves that the area of the circle Z, and the area of a circumscribing square to the circle M, are exactly equal.

Well, then, by means of the enclosed diagram, I have proved that a square equal to a given circle, can be constructed, isolated, and exhibited ; and in a way that no Geometer can fail to perceive, or any candid Mathematician hesitate to admit. Other proofs are almost self-evident.

Now, my dear Sir, you will see that I have "*planted*" some very "*hard hits*" in this round, and I cannot help thinking that Mr. R——'s second (your brother) will have to "*throw up the sponge*," and admit that Mr. R—— is thoroughly and fairly beaten. You will observe from the diagram, that I have indicated another triangle by dotted lines, but have not, as yet, made any use of it. Should Mr. R—— make up his mind for another round, I shall "*plant*" some other "*hard hits*" by means of it.

Your favour of the 7th inst. is to hand, which I shall reply to in a separate Letter, and with kind regards to you and yours,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

MR. JAMES SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH,
13*th April*, 1868.

MY DEAR SIR,

Your brother's packet of Papers, enclosed in your favour of the 7th inst., has greatly amused me. I have often been *twitted* by Mathematicians, for suggesting the idea of such a thing, as the possibility of a proof of the question at issue between us, by weight. Your brother is the first Mathematician who has ever *hinted* at such a means of proof.

Well, then, some 7 or 8 years ago, it occured to me and a Mechanical friend—who is a practical Geometer, but, who makes no pretensions to be a Mathematician—and—you know that a man may be "*ignorant of even the Multiplication Table, and yet be able to master all the propositions of Euclid*"—that, if the *theory*, that 8 circumferences of a circle are exactly equal to 25 diameters, *be true*, it may be proved by weight.

We went to work, first with cardboard, then with block tin, then with rolled copper, weighing the circle against its equivalent parallelograms and square, and found them an exact balance. Encouraged by our success, we then adopted the following method of proof, which does not involve any consideration of the diameter of the circle; since, if the diameter of a circle be divided into 16 equal parts, it follows of necessity, that if 8 circumferences=25 diameters, a parallelogram of which the shorter sides = 8 and the longer sides = 25 of these parts, will contain the same superficial area, and be of equal weight with the circle, whether these figures be constructed in gold, silver, copper, lead, or iron, so that they are of the same density and thickness.

Well, then, we had a plate cast in iron, taking the greatest care that it should be a perfect casting, so as to be assured of its being the same density throughout. We then had this plate of iron planed on both sides by the very best machinery, so as to be assured of its being of the same thickness throughout. We then had one part of this iron plate carefully turned into a circular disc by the very best machinery, without reference to any particular diameter, and the mechanic who attended to the construction, had not even an idea of what was our object. The disc was made as large as the plate would allow, and was about 17 or 18 inches diameter, and one inch in thickness. Having got this circular disc, we ascertained its exact diameter by a slender steel rod, and dividing this rod into 16 equal parts, we then had made from another part of the plate, a parallelogram, of which the shorter sides were equal to 8, and the longer sides equal to 25 of these parts, and on comparing them in carefully adjusted balances, we found the circular disc and parallelogram to be of exactly equal weight. From your brother's mechanical turn of mind, he will comprehend this mode of proof without any difficulty, and he cannot fail to perceive that it is much more practical, and more accurate, than his supposed proof by weight, by mere tissue paper.

I enclose a geometrical study for your brother's especial edification. This geometrical figure is a fac-simile of that enclosed in my letter of Saturday, with the following additions.

From the point F, draw a straight line at right angles to O F, and therefore tangental to the circle M, to meet the line O E produced at the point R, constructing the right-angled triangle O F R. With O as centre, and O R as interval, describe the circle X Z. From the point R,

draw a straight line at right angles to O R, and therefore tangental to the circle X Z, to meet O F produced at the point V, constructing the right-angled triangle O R V. N T and L E, the diameters of the circle Z, intersect the circumference of the circle M at the points *m*, *n*, *o*, and *p*; and also intersect the circumference of the circle X at the points *a*, *b*, *c*, and *d*. Join *m n*, *n o*, *o p*, and *p m*, producing the inscribed square *m n o p* to the circle M; and join *a b*, *b c*, *c d*, and *d a*, producing the inscribed square *a b c d* to the circle X. About the circle P, circumscribe the square *a′ b′ c′ d′*.

If Mr. R—— should have the opportunity of seeing this peculiar and remarkable geometrical figure, it will—or at any rate it ought to—give him a "*wrinkle*" as to the "*hard hits*" he may expect in our next "*round*." With best wishes,

Believe me, my dear Sir,
Very truly yours,
JAMES SMITH.

J—— S——, Esq.

Mr. JAMES SMITH *to* Mr. S——.

BARKELEY HOUSE, SEAFORTH,
14th April, 1868.

MY DEAR SIR,

I shall probably not post what I am about to write, until I hear what Mr. R—— has to say, in reply to my Letter of the 11th inst. Not having anything to call me to Liverpool to-day, I shall proceed to prepare a Letter, directing your attention to some most remarkable geometrical and mathematical truths, demonstrable beyond the possibility of dispute or cavil—

by any candid Geometer and Mathematician—by means of the geometrical figure contained in my Letter of yesterday. *(See Diagram IX.)*

I may remind you, that your favour of the 2nd March brought me a long, and—to me—extraordinary Paper of Mr. R——'s, which was very nearly bringing the controversy between me and that gentleman to a close, and would certainly have done so, but for certain admissions made by him in a subsequent Paper. Now, before proceeding to the more immediate object of this Letter, I must refer to some of Mr. R——'s statements, arguments, and conclusions, in this remarkable Paper, which commences thus:—"*In his Paper of the* 15*th (Feby.), Mr. Smith revealed the whole process by which he had arrived at* $\pi = 3\frac{1}{8}$." Now, my dear Sir, it will be obvious enough to you, that I have not, even yet, revealed the *whole* process by which I have arrived at the fact—*for fact it is*—that $\pi = 3\frac{1}{8}$. How, then, was it possible that I could reveal the *whole* process by which I arrive at this conclusion, in a single Letter? It is no doubt true, that in my Letter of the 15th February, I—honestly and candidly—revealed the first steps in my process, viz., when the circumference of a circle $= 4$, the diameter and area are represented by the same arithmetical symbols: and, when the diameter of a circle $= 4$, the circumference and area are represented by the same arithmetical symbols, whatever be the arithmetical value of π. These facts may be demonstrated by means of any *finite* hypothetical value of π, intermediate between 3 and 4; that is, intermediate between the perimeter of a regular inscribed hexagon, and the perimeter of a circumscribing square, to a circle of which the diameter is unity $= 1$. Making

this discovery, I was led to search for a *finite* value of π —and what other motive has any Mathematician for trying his hand at circle-squaring, but the *apparent irrationality* of supposing π to be an *indeterminate* arithmetical quantity—and found one.

Mr. R—— next observes:—"*In my last I showed that this process proceeds on the* PRINCIPLE *that* π IS *determinate. To this I called Mr. Smith's attention, and requested a proof of a premiss so fundamental to his conclusion. If he cannot show satisfactorily that this premiss is true, what right has he to insist that I or any man admit his conclusion? Now, in his Letter of the 22nd, Mr. Smith takes no notice of this—says not a syllable about it; and what conclusion am I to draw from this reticence?*" Now, I might appeal to any man possessed of moderate geometrical and mathematical attainments, to say, whether I have not, in my Letter of the 22nd February, proved in several ways—among others, by examples of continued proportion—that $\pi = 3\frac{1}{8}$, and is therefore *determinate.* How Mr. R—— could charge me with *reticence*, is *still* to me a mystery. Can Mr. R—— show, by any of my Letters, a desire on my part to evade fair argument or logical reasoning?

Again: Mr. R—— says:—"*Mr. Smith complains that I charge him with asserting that the number* 4 *cannot be divided by any number but* $3\frac{1}{8}$ *and give a determinate quotient: and asks where he ever made such an assertion. To say that* $3\frac{1}{8} \times 1\frac{7}{25}$ *are the only arithmetical symbols that will produce* 4 *is the same as to say, that* $3\frac{1}{8}$ *is the only number that will divide* 4 *and give a determinate quotient. To me, these seem identical expressions. I cannot see how Mr. Smith can question their identity. If they are not identical,*

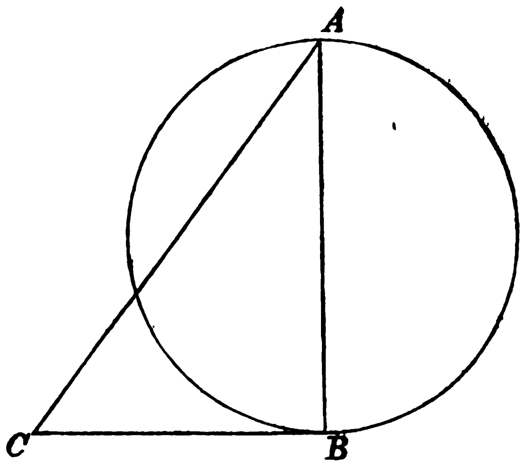
then I have misrepresented the proposition: but if Mr. Smith cannot show that these two propositions are not identical, in asserting the one he asserts the other; and it is perfectly legitimate for me to substitute the one form for the other." Mr. R—— then triumphantly puts the question:—"*Is it fair in him to complain of this?*" Now, in my Letter of the 15th February, having shown that in the analogy or proportion, $a : b : \; b : c$, if $a = \frac{\pi}{4}$ and $b = 1$, then, $c =$ diameter of a circle of which the circumference is 4, whatever be the value of π; and, that on the *theory* that 8 circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3{\cdot}125$, the value of π, c is a finite quantity $= 1{\cdot}28$; and, that $3(1{\cdot}28) = 3{\cdot}84$ is the perimeter of a regular inscribed hexagon to a circle of diameter 1·28. I then drew the following conclusions: "*Now, since no other arithmetical symbols but* $3{\cdot}125 \times 1{\cdot}28$ *will produce* 4, *it follows of necessity, that* 3·125 *must be the true value of* π: *and* $\frac{24}{25}$ *(circumference)* $= \frac{24}{25}(4) = 3{\cdot}84$, *the perimeter of a regular inscribed hexagon to a circle, when the circumference* = 4." Was it *fair* of Mr. R—— to catch at, and play with, the former part of this quotation, and treat with contempt the latter part? Do not the two conclusions necessarily stand or fall together? You cannot fail to perceive, my dear Sir, that Mr. R——'s reasoning (can it be called reasoning?) would actually make the perimeter of a regular inscribed hexagon to a circle of diameter = 1·28, an *indeterminate* quantity. In my Letter of the 22nd February, I called Mr. R——'s attention to the fact, that Orthodoxy "*makes* π *an indeterminate decimal* = 3·14159265..., *said to be true to these* 8 *places of decimals*," and that this value of π is within the limits of $3\frac{1}{8}$ and $3\frac{1}{7}$; and from this fact

drew a conclusion with reference to Mr. R——'s 5 pairs of numbers. He does not dispute (how could he?) that the orthodox value of π is within the limits of $3\frac{1}{8}$ and $3\frac{1}{7}$. Now, it appears to me inconceivable, that when Mr. R—— penned this extraordinary Paper, *he did not know*, that when I said "$3\frac{1}{8} \times 1\frac{7}{25}$ *are the only arithmetical symbols that will produce* 4, the words "*within admissible limits*" were implied, it being admitted by both of us, that when the circumference of a circle $= 4$, the diameter and area are expressed by the same arithmetical symbols. Well, then, it seems to me, that Mr. R—— has no wish to reason out to a true conclusion the question at issue between us, and therefore resorts to catching at, and playing upon words, and even perverting my meaning, if I happen to have so expressed myself, as to give him the slightest chance of doing so.

Again: My Letters of the 15th and 17th of February should be taken together. In the latter, by some slight additions to a diagram contained in the former, I gave a remarkable proof of the true value of π. The facts to which I directed Mr. R——'s attention by the geometrical figures represented by these diagrams, he has never condescended to notice in any way whatever. Can this be called *fair* controversy? But, never mind. I challenge him to fairly "*pick a hole in my coat.*" I have already *picked many holes in his*, and have some to pick yet. Well, then, in another part of his remarkable Paper, on my Letter of the 15th February, he puts the question: "*Are not* $3\frac{4}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, *and* $3\frac{1}{3}$, *all as determinate as* $3\frac{1}{7}$ *and* $3\frac{1}{8}$*?*" I emphatically answer, No!! The former fractions cannot be expressed in finite decimals: the latter can! Hence: We can prove the true ratio of diameter to cir-

cumference in a circle, by either of the latter fractions: but, can we do so by any of the former fractions? Certainly not!! Now, my dear Sir, as referee in the contest between me and your relatives, you will say, I should let them have the proofs of these startling assertions, and you are right! Well, then, they shall have the proofs. Here they are!

The fraction $3\frac{1}{8}$, expressed decimally, $= 3{\cdot}125$. Hence: $\frac{3{\cdot}125}{4} : 1 :: 1 : 1{\cdot}28$; that is, $\cdot78125 : 1 :: 1 : 1{\cdot}28$; therefore, $\frac{3{\cdot}125}{1}$ and $\frac{4}{1{\cdot}28}$, are equivalent ratios, and both express the true ratio of circumference to diameter in a circle. But, the fraction $3\frac{1}{5}$, expressed decimally, $=$ $3{\cdot}2$. Hence: $\frac{3{\cdot}2}{4} : 1 :: 1 : 1{\cdot}25$; that is, $\cdot8 : 1 :: 1 : 1{\cdot}25$; therefore, $\frac{3{\cdot}2}{1}$ and $\frac{4}{1{\cdot}25}$, are equivalent ratios. Now, $3(1{\cdot}25)$ $= 3{\cdot}75$; and $3{\cdot}75 =$ the perimeter of a regular inscribed hexagon to a circle of diameter $1{\cdot}25$; therefore, $\frac{3{\cdot}125}{3}$ $(3{\cdot}75) = \frac{3{\cdot}125 \times 3{\cdot}75}{3} = \frac{11{\cdot}71875}{3} = 3{\cdot}90625$; therefore, $3{\cdot}90625 : 1{\cdot}25 :: 3{\cdot}125 : 1$. Hence: $\frac{3{\cdot}90625}{1{\cdot}25}$ and $\frac{3{\cdot}125}{1}$ are equivalent ratios, and both express the true ratio of circumference to diameter in every circle. Pray, what becomes of Mr. R——'s fractions $3\frac{1}{7}$, $3\frac{5}{9}$, $3\frac{7}{11}$, and $3\frac{1}{3}$? Can he work out a similar result by any of them? Can Mr. R—— find any other fractions but $3\frac{1}{5}$ and $3\frac{1}{8}$, by which he can work out equivalent ratios? I trow not!! Well, then, this is simply an effect; but there cannot be an effect without a cause. Now, let us try if we cannot trace this effect to its true cause.



In this simple geometrical figure, A B is the diameter of the circle, and A B C is a right-angled triangle, of which the sides A B and B C, which contain the right angle, are in the ratio of 4 to 3, by construction.

Now, if $AB = 1$, then, $\frac{3}{4}(AB) = \frac{3}{4}(1) = \cdot 75 = BC$; therefore, $(AB^2 + BC^2) = (1^2 + \cdot 75^2) = (1 + \cdot 5625) = 1\cdot 5625 = AC^2$; therefore, $\sqrt{AC^2} = \sqrt{1\cdot 5625} = 1\cdot 25 = AC$. Hence: $AC = 5$ times the semi-radius of the circle; and, if X denote semi-radius of the circle, then, it follows of necessity, that $5(X) : 1\cdot 25 :: 4(X) : 1$; therefore, $\frac{\pi(AC)}{5X}$ and $\frac{\pi(AB)}{4X}$, are equivalent ratios, and both express the true ratio of circumference to diameter in every circle. But further: $(AB^2 + BC^2 + AC^2) = \{(AB + BC)^2 + (AB - BC)^2\}$, and this equation $= 4X$. Hence: The area of every circle is equal to the sum of the areas of squares about a right-angled triangle, of which the sides that contain the right angle are in the ratio of 4 to 3, and the longer of these sides the radius of the circle. Corollary: The area of every circle is equal to the area of a square on the hypothenuse of a right-angled triangle, of which the sides that contain the right angle are in the

ratio of 7 to 1, and the sum of these two sides the diameter of the circle.

Again: When $AB = 1$, one-fifth part of the circumference of the circle $= \frac{AC}{2}$, that is, $\frac{3\cdot125}{5} = \frac{1\cdot25}{2}$, and this equation $= \cdot625$; therefore, $\left(\frac{3\cdot125}{5}\right)^2 = \left(\frac{AC}{2}\right)^2$, and this equation $= \cdot390625$; therefore, $\left(\frac{3\cdot125}{5}\right)^2 = \frac{3\frac{1}{8}(AB)^2}{2}$ and this equation $= \cdot390625$, that is, $=$ half the area of the circle. Hence: one-fifth part of the circumference of any circle $=$ side of a square containing half the area of the circle. Corollary: One-fifth part of the perimeter of any square $=$ diameter of a circle containing half the area of the square.

Now, my dear Sir, have I not in previous Letters established these truths by practical Geometry? Will Mr. R—— *dare* to tell me that the equation $(AB^2 + BC^2 + AC^2) = \{(AB + BC)^2 + (AB - BC)^2\}$ is not equal to $3\frac{1}{8}(AB^2)$? Will he *dare* to tell me that πr^2 is not equal to area in every circle? I trow not! Well, then, all the facts, to which I have directed your attention, follow of necessity, and establish the truth of the *theory* that 8 circumferences $=$ 25 diameters in every circle, which makes $\frac{25}{8} = 3\cdot125$, the true arithmetical value of π.

Now, these truths dispose of all the ***nonsensical*** assertions contained in the following remarkable paragraphs from Mr. R——'s extraordinary Paper, in reply to my Letter of the 15th February.

"*I now come to his complaint that I do not grapple with his proofs by practical Geometry. Why, it was my grappling with one of them, that sent to the Duke of*

Buccleuch that brought us to this pass. More than this, I have declared my readiness to grapple with the whole of them. But it is far more to the point to go to the first principles with which Mr. Smith proceeds. In his last, he says he assumes the THEORY *that* $\pi = 3\frac{1}{8}$, (I said I assume a THEORY which makes $\pi = 3\frac{1}{8}$,) *and proves it in many ways by practical Geometry. But is this consistent with the profession he made in a previous communication, to have revealed the whole process of reasoning by which he proved* $\pi = 3\frac{1}{8}$. (I have never yet said that I had revealed the *whole* process of reasoning by which I prove $\pi = 3\frac{1}{8}$.) *If a thing is proved by a process of reasoning, how can it be assumed as a* THEORY? (The *theory* I assumed is, that 8 circumferences = 25 diameters in every circle, and maintain that the truth of this *theory* can be proved by practical Geometry.) *Mathematics admit of no assumed theories.* (This assertion I have disposed of in a previous Letter.) *Everything must be proved, rigorously from the axioms or identical propositions, or no assent can be demanded.*" (Granted.)

"*But I further assert, that if Mr. Smith means by practical Geometry what every other person means, he can prove nothing by it. Practical Geometry is merely the application to the construction of figures; or the doing of something with lines or figures, which* THEORETICAL *Geometry has already shewn us the way to do. Practical Geometry is to* THEORETICAL *what the rules of arithmetic are to the higher and far more interesting and recondite analysis by which they are derived. None of Mr. Smith's geometrics is a proof that* $\pi = 3\frac{1}{8}$. *In every one there is the impassable hiatus.*"

"*Take the very last. On page* 10, *referring to the*

figure (See Diagram I.), Mr. Smith says:—'If Mr. R—— knows anything of practical Geometry, and chooses to apply his knowledge, he may convince himself that square of E C, *that is,* ($\frac{3}{8}$ *diameter*)^2 = *half the area of the circle* Y. *This is mere trifling with me. It is to say that I can prove the very thing I ask Mr. Smith to prove; and if I cannot, or will not, that by implication I know nothing of Geometry. If Mr. Smith could prove this himself, he would have done it long ago."*

"Now the sum of the whole matter is, that the reasoning by which Mr. Smith arrived at his π *is fundamentally wrong, unless he has something in reserve. But as I asked this in my last, and nothing has come, I conclude that there is nothing behind. The process therefore is a failure.* 2. *The assumed theory has not been proved by practical Geometry, and cannot be proved by it. It must be proved by theoretical Geometry. I know enough of Geometry to understand anything in the shape of proof that Mr. Smith can offer. Indeed, my chief difficulty all along has been the confident way in which he has asserted the very thing that requires rigorous proof. I am ready to justify this complaint by analysing the other modes with which Mr. Smith has favoured us."*

The division of the diameter by 7 : 1, *produces certain results. For instance, the differences of the squares of these parts = the inscribed dodecagon. But this has no connection whatever with* π. (Indeed, Mr. R——! has πr^2 no connection with the circle?) *The real nexus between this division and* π *is, that the sum of these squares = the area of the circle; and this Mr. Smith asserts, not proves. Were* $3\frac{1}{8} = \pi$, *then the square = the circle would thus be constructed and exhibited. But that does not*

prove that $\pi = 3\frac{1}{8}$. *Surely Mr. Smith will not say that it does.* (Indeed, Mr. R—— ! Mr. Smith does say that it proves $\pi = 3\frac{1}{8}$. If, with $\pi = 3\frac{1}{8}$, a square equal to a given circle can be constructed and exhibited—and that such square exists is admitted by Geometers—what else can the value of π be but $3\frac{1}{8}$? If the diameter of a circle $= 8$, will Mr. R—— tell me that $3\frac{1}{8}$ (r^2) is not equal to $7^2 + 1^2$?)

In another paragraph Mr. R—— says :—" *He (Mr. Smith) asserts that* $\frac{24}{25}$ *(circumference* $= 6r$ *or* $3d =$ *the perimeter of a regular inscribed hexagon, and I see this assertion repeated in his Letter to Mr. Jno. S—— with the same confidence as before, and myself named as wanting the candour to admit it." This is simply power of assertion, not force of reasoning. If again:* π *were* $3\frac{1}{8}$, *this would be true: or if this could be shown to be true independently of* π, *it would follow that* $\pi = 3\frac{1}{8}$. *But to use the former to prove the latter, and then the latter to prove the former, is simply astounding: it is not reasoning at all.*"

I candidly confess, that when I read this paragraph, I was astounded. Now, Mr. R—— admits that if π were $3\frac{1}{8}$ it would be true, that $\frac{24}{25}$ (circumference) $= 6$ times radius, or 3 times diameter; that is, $=$ the perimeter of a regular inscribed hexagon to the circle: and, we are agreed that when the circumference of a circle $= 4$, the diameter and area are represented by the same arithmetical symbols. Well, then, Mr. R—— will not venture to dispute that the *diameter* of a circle may be represented by the arithmetical symbols 1·28; and if so represented, then, $3\ (d) = 3·84 =$ the perimeter of an inscribed regular hexagon to

the circle. Then: $\frac{3 \cdot 125}{3}(3 \cdot 84) = \frac{3 \cdot 125 \times 3 \cdot 84}{3} = \frac{12}{3}$ $= 4$. "*π does not appear, need not appear.*" But, this proves that what is represented by the symbol π in mathematical phraseology $= 3 \cdot 125$, that is, $= 3\frac{1}{8}$. Again: Mr. R—— will not venture to dispute that the diameter of a circle may be represented by the arithmetical symbols 1·25, and if so represented, then, $3\,(d) = 3 \cdot 75 =$ perimeter of a regular inscribed hexagon to the circle. Then: $\frac{3 \cdot 125}{3}(3 \cdot 75) = \frac{3 \cdot 125 \times 3 \cdot 75}{3} = \frac{11 \cdot 71875}{3} = 3 \cdot 90625$. "*$\pi$ does not appear, need not appear.*" But, $\frac{3 \cdot 90625}{1 \cdot 25}$ and $\frac{4}{1 \cdot 28}$ are equivalent ratios, and proves that what is represented by the symbol π in mathematical phraseology $= 3 \cdot 125$, that is $= 3\frac{1}{8}$; or in other words, that the diameter is contained exactly $3\frac{1}{8}$ times in the circumference of every circle. Mr. R—— may controvert this proof, if he can find a value of π, by which he can make $\frac{\pi}{3}\ (3 \cdot 84) = 4$. Let him try, and where will he be? Certainly not in π glory!! Then, my dear Sir, what can the arithmetical value of π be but $3\frac{1}{8}$; and what can $\frac{24}{25}$ (circumference) be equal to in any circle, but the perimeter of a regular inscribed hexagon? And what can $\frac{4}{25}$ (circumference) be in any circle, but an arc equal to radius?

The last paragraph in Mr. R——'s extraordinary Paper runs as follows:—"*Glancing at the mathematical demonstration by continued proportion, I have only to say that it demonstrates nothing but that* 9·765625 = *the* 5*th term in a series of which the first term is* 4, *and the common ratio* $\frac{5}{4}$

To offer this as a PROOF *that* $3\frac{1}{8} = \pi$, *is playing with me.*" This appears to me like a man *glancing* at a distant mountain, and who, either from defective vision, or want of reflection, takes it for a cloud. Mr. R—— had just before said :—"*The assumed theory has not been proved by practical geometry, and cannot be proved by it.*" (Indeed, Mr. R——, we shall see !) He followed this up by saying:—"*It must be proved, if done, by theoretical geometry.*" (I venture to tell Mr. R—— that it can be proved by practical geometry, and demonstrated to be in perfect harmony with theoretical geometry.) Mr. R—— then says :—"*I know enough of geometry to understand anything in the shape of proof that Mr. Smith can offer.*" Very well, Mr. R——: I am glad to hear it. For, if this assertion be true, we shall be certain to arrive in the end, at "*a happy state of concord*" as to the value of π ; or in other words, as to the true ratio of diameter to circumference in a circle.

Well then, Mr. R—— *asserts* that the mathematical demonstration by continued proportion, which I gave him in my Letter of the 22nd February, "*proves nothing,*" and that to offer it as a proof that $\pi = 3\frac{1}{8}$, is "*playing with him.*" These are bold assertions, and I will now shew you, my dear Sir, that they utterly fail, when tried by geometrical and mathematical evidence.

Now, referring to the enclosed diagram *(see Diagram IX.)*, the triangles O B C and O B P are similar and equal right-angled triangles, by construction. And O C and O P the sides subtending the right angles, are the perpendiculars of the triangles O C F and O P F, by construction. But, C B and P B are straight lines drawn from the right angles perpendicular to their opposite sides, in the triangles O C F and O P F ; and O F is the side subtending the

right angles C and P in the triangles O C F and O P F, and common to both ; therefore, O B C and C B F are similar triangles, and similar to the whole triangle O C F : and similarly : O B P and P B F are similar triangles, and similar to the whole triangle O P F. Euclid, Prop. 8, Book 6. Again : For the same reasons, O C F and F C R are similar triangles and similar to the whole triangle O F R. Again : For the same reasons, the triangles O F R and R F V are similar triangles, and similar to the whole triangle O R V. It is self-evident, that we might go on multiplying similar triangles, *ad infinitum.* Then : Is it not self-evident, that theoretical and practical geometry go hand in hand ?

Well, then, from this piece of practical geometry, that is, from the enclosed diagram, we get the following example of continued proportion, all the symbols of which represent straight lines, the lines themselves being radii of circles. I have not thought it necessary to describe the circle, of which O V would be the radius.

O B : O C :: O C : O F :: O F : O R :: O R : O V.

Then :

When O B = 4 ; O C = 5 ; O F = 6·25 ; O R = 7·8125 ; and O V = 9·765625 ; therefore, O V = $3{\cdot}125^2$.

And :

When O B = ·5 ; O C = ·625 ; O F = ·78125 ; O R = ·9765625 ; and OV = ·1220703125 ; therefore, OV = $\frac{\left(\frac{3\cdot125}{2}\right)^2}{20}$, and, $\frac{O\,R^2}{O\,F} = 10\left(\frac{3\cdot125}{2}\right)^2 = 10\,(O\,V)$. The terms of the series are in the ratio of $\frac{5}{4}$, and are inseperably connected with the similar right-angled triangles O B C, C B F, O C F, F C R, O F R, F R V, and O R V.

Now, one thing Mr. R——. has admitted, viz.: that the circumference of circles are to each other as their radii. Well, then, O B is the radius of the circle X; O C is the radius of the circle Y; O F is the radius of the circle M; O R is the radius of the circle X Z; and O V the radius of another circle which Mr. R—— may describe for himself. Now, that gentleman may calculate the circumference of these circles by any value of π he pleases, intermediate between 3 and 4. It may be determinate if he like, or indeterminate if he prefer it, and can work out the calculations by such a value of π. The ratio of circumference to circumference, will be the same as the ratio of radius to radius in any two of these circles. Mr. R—— may "*Stick to Algebra*" if he pleases, but I defy him to controvert the facts. Well then, under these circumstances the question arises:—" How are we to detect the true value of π from all counterfeits? The answer appears to me plain and simple, viz.:—By connecting it with some other figures in the diagram, and producng results to which every other but the true value of π is utterly incompetent. Now, Mr. R—— has admitted another thing, viz.: π (r^2) = area in every circle. Well then, on the *theory* that 8 circumferences = 25 diameters in every circle.

$3\frac{1}{8}$ $(O\,B^2)$ = the sum of the areas of squares about the triangle O B C.

$3\frac{1}{8}$ $(O\,C^2)$ = the sum of the areas of squares about the triangle O C F.

$3\frac{1}{8}$ $(O\,F^2)$ = the sum of the areas of squares about the triangle O F R.

And, $3\frac{1}{8}$ $(O\,R^2)$ = the sum of the areas of squares about the triangle O R V.

Again: *a b c d* is an inscribed square to the circle X, and a circumscribing square to the circle X Y, and *d c* a side of it is bisected by the line O *v*, for *d c* and O *v* are diagonals of the square O *c v d*. But O *e* = O *c*, for they are radii of the circle X; therefore, the square O *e f g* = the square O*cvd*. Now, O A, O B, O*c*, O*e*, O*d*, and O*g* are equal, for they are radii of the circle X. Take one of these radii, say O B. Then: If O B = 4, $4(\text{O B}^2) = 4(4^2) = 64$ = area of a circumscribing square to the circle X; and the area of an inscribed square = half the area of a circumscribing square in every circle; therefore, $\frac{64}{2} = 32$ = area of the square *a b c d*, and *a b c d* is a circumscribing square to the circle X Y. But, O *t d h* is a square on the radius of the circle X Y; therefore, $\frac{32}{4} = 8$ = area of the square O *t d h*; therefore, O *t* a radius of the circle X Y = $\sqrt{8}$, and $\pi (r^2)$ = area in every circle; therefore, $\pi (\text{O } t)^2 = 3{\cdot}125 \times 8 = 25$ = area of the circle X Y, and is equal to half the area of the circle X, and twice the area of the circle P. Hence: an inscribed square to the circle X is a circumscribing square to the circle X Y; and an inscribed square to the circle X Y is a circumscribing square to the circle P. Now, it is obvious that about the circle X we may construct a circumscribing square, and about the square describe a circle, and so on *ad infinitum;* and to whatever extent we might carry this operation, if the side of the first square be represented by a finite quantity, the diameter of every alternate circle and side of its circumscribing square would be a finite quantity; and the diameters of the other alternate circles and the sides of their circumscribing squares, the square root of a finite quantity. But, if π

were indeterminate, the areas of all the circles would be also indeterminate, while the areas of all the squares would be determinate, and no definite ratio could exist between the areas of the circles and the areas of their circumscribing squares, which is not only irrational but absurd. We know that the area of every circle and of every square is the double of that which precedes it, then how in the name of common sense can the area of the circle be indeterminate? Well then, the Mathematician may as well attempt to prove, that the point t is not the centre of the square O $c\,v\,d$ in the diagram, or that O $d\,e\,c$ is not a quadrant of the circle X, as attempt to prove that the symbol π does not represent a determinate arithmetical quantity. Mr. R—— will not venture to tell me that we must first find the value of π before we can prove that the area of a circumscribing square = twice the area of an inscribed square in every circle.

Again: O F is the radius of the circle M, and when O B, the radius of the circle X, $= 4$, O F $= 6{\cdot}25$. Now, $4\,(\text{O F}^2) = 4\,(6{\cdot}25^2) = 4 \times 39{\cdot}0625 = 156{\cdot}25 =$ area of a circumscribing square to the circle M. But $m\,n\,o\,p$ is an inscribed square to the circle M; therefore, $\frac{156{\cdot}25}{2} = 78{\cdot}125 =$ area of the square $m\,n\,o\,p$. Now, O C is the radius of the circle Y, and when O B $= 4$, O C $= 5$, and C F $= \frac{3}{4}(\text{O C})$ by construction; therefore, $\frac{3}{4}(\text{O C}) = \frac{3}{4}(5) = 3{\cdot}75 =$ C F; therefore, $12{\cdot}5\left(\frac{\text{O C}}{2}\right)^2 = (\text{O C}^2 + \text{C F}^2 + \text{O F}^2)$, that is, $12{\cdot}5\,(2{\cdot}5^2) = (5^2 + 3{\cdot}75^2 + 6{\cdot}25^2)$, and this equation $= 78{\cdot}125 =$ area of the square $m\,n\,o\,p$. But, $3\frac{1}{8}\,(\text{O C}^2) = (\text{O C}^2 + \text{C F}^2 + \text{O F}^2)$, and this equation $= 78{\cdot}125 =$ area of the circle Y, on the *theory* that 8

circumferences = 25 diameters in every circle, which makes $\frac{25}{8} = 3\cdot125$ the arithmetical value of π. Hence: on this *theory*, the circle Y and the square $m\,n\,o\,p$ are exactly equal in superficial area.

Now, about the square $m\,n\,o\,p$ we may describe a circle, and about the circle construct a circumscribing square, and so on we might proceed, *ad infinitum*. And about the circle Y we may construct a circumscribing square, and about the square circumscribe a circle, and so on we might proceed, *ad infinitum*. Let Mr. R—— reflect upon the absurdity of supposing that every successive circle and square, in both cases, will be the double of that which precedes it, and yet, that the areas of all the circles shall be indeterminate arithmetical quantities. If this is not self-evident, Mr. R—— has only to put a finite value upon the area of the first circle, that is, the circle Y, and he will make the diameters of every circle and the areas of every square indeterminate, in working out the calculations with his indeterminate π.

I had written so far when Miss S——'s favour, enclosing Mr. R——'s rejoinder to my Letter of the 7th inst. came to hand. I shall reply to it in a separate Letter. Mrs. Smith and I regret to hear that you are suffering from an attack of inflammation, but hope to hear of your speedy recovery, and with our united kind regards to yourself, Mrs. and Miss S——.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. R——'s Papers, *April 8th and 16th*, 1868.

(One Communication.)

Unless first principles are well established—proved beyond question, if not axiomatic or self-evident—no discussion can be worth anything, or possess the least interest to sincere and intelligent men. I am perfectly sure that Mr. Smith will admit, that if π cannot be shown to be a determinate quantity, and shown by *a priori* reasoning, *i.e.*, without reference to its arithmetical value—that process of reasoning by which he some time ago said he arrived at his conviction that $\pi = 3\frac{1}{8}$ cannot be valid. That π is determinate is, I say, a first principle in that process. Now, I again question the truth of that princple—or, rather, proposition. As frequently I have said, it is not self-evident; and I know of no way in which it can be proved *a priori*. Surely it cannot be proved by practical geometry, or by calculations. It must be established by some kind of *a priori* and abstract reasoning: because, it is brought in by Mr. Smith to find the arithmetical value of π. Now, I humbly submit that all Mr. Smith can say is away from the point, until he meet this claim I again make, viz. that he show how we must believe π to be a determinate quantity.*

* Does not the argument adopted by Mr. R—— in this paragraph amount to this:—We must find the arithmetical value of π, without the aid of arithmetic! What other interpretation can be put upon the following sentence? "I am perfectly sure Mr. Smith will admit, that if π cannot be shown to be a determinate quantity, and shown by *a priori* reasoning, *i.e.*, without reference to its arithmetical value—that process of reasoning by which he, some time ago, said he arrived at his conviction that $\pi = 3\frac{1}{8}$, cannot be valid." *How are we to know whether π be arithmetically determinate or indeterminate, until by some process or other we have found an arithmetical value of it?*

In the long paper, dated 1st April, which I have read through, I see nothing that faces this initial difficulty. There is a reference to this matter on p. 19, where Mr. S—— makes me to say, "Mr. H—— does not make π determinate." Mr. Smith's understanding of Mr. H——'s meaning may be correct. But it is best to hear a man say himself what he means, unless it be so plain as to be beyond doubt, which is not the case with Mr. H——'s oracle. But even were it that Mr. H—— means to say that his π is determinate, Mr. Smith must allow that unless we see Mr. H——'s *process*—how he gets his π, no man's faith in it can be demanded.

On p. 28 Mr. Smith repeats a form of statement which occurs all through, and which I shall examine. "Since no other value of π than $3\frac{1}{8}$ will fulfil these conditions, it follows that $3\frac{1}{8}$ must be $= \pi$;" and he then puts the question to me: "How in the name of common sense can π be otherwise than a determinate arithmetical quantity?" Now, I ask: how in the name of common sense does Mr. Smith think this meets my objection? Can Mr. Smith not see that this is reasoning in a circle? He told us that by a process of reasoning from the principle that π is determinate, he found that it must be $3\frac{1}{8}$; and here, when asked to prove the fundamental principle true, he says, see—π is $3\frac{1}{8}$, how can it be otherwise than determinate? Is this all that he has to say in proof of this point, so fundamental, and which played so important a part in his original investigation? I find the same statements repeated, which I have repeatedly questioned, and shown to need proof—but I find them repeated with the same air of confidence: as for instance, that the perimeter of a regular hexagon $= \frac{24}{25}$ (circumference.) This is wearisome. Unless Mr. Smith go deeper, and look at the foundation, this wearisome reiteration, without proof, will merely convince us of the opposite of his convictions.

April 16.

Since writing the above two other Papers have been received. In the last one, dated 7th April, there is an attempt

to meet the fundamental difficulty above alluded to. Since $x : z :: 16 : 50 :: 1 : 3\frac{1}{8}$; or, since $z = 3\frac{1}{8}(x)$, when the diameters of these circles are 8 and 10 ($\sqrt{2}$) respectively, Mr. Smith asks, how π can be *indeterminate?* One can only reply: it does not matter what π is: it is the same in both x and z, and nothing as to its character or value can be inferred from this ratio. Suppose it were $3\sqrt{2}$, a surd quantity; and, r and r' the radii, respectively, $3\frac{1}{8}\ \pi\ r^2 = \pi r'^2$, or, $3\sqrt{2}\ r'^2 = 3\frac{1}{8} \times 3\sqrt{2} \cdot r^2$. The $3\frac{1}{8}$ is not π, but the ratio of the one area to the other area.

On the second page of this Letter Mr. Smith puts me in a dilemma. If (a) be area of circle, and (b) the side of circumscribing square, *i.e*, diameter, then $\frac{a}{\frac{1}{4}\pi} = b^2$. "And it follows, that if π be indeterminate, b must be indeterminate also;" and he actually asserts that my reasoning implies "that the *diameter* of a circle inscribed in a square, cannot be arithmetically expressible by any finite number; or, by the square root of any finite number." Now, the side of the square is the diameter of the circle, and if that side be determinate, the diameter of the circle, being the *same*, must, of course, be determinate also. There is no dilemma here—not the shadow of a horn. I would simply say that, in such a case, (a) the *area* is indeterminate, but being divided by π, the quotient, $\frac{a}{\frac{1}{4}\pi}$ will of course give you back b, after the little shuffle of the cards. $a = \frac{1}{4}\ \pi\ b^2$ and $\frac{a}{\frac{1}{4}\pi} = \frac{\frac{1}{4}\pi\ b^2}{\frac{1}{4}\pi} = b^2$. This is surely plain enough, without any more figuring.

The next paragraph on page 2 is not intelligible in its present form. $z = 3\frac{1}{8}\,x$: and I asked, "how this upset the fact that π is indeterminate?" Mr. Smith replies that in "*assuming* it to be a *fact* that π is indeterminate, I might as well assume that the area of a square is indeterminate. Could absurdity go further?" What absurdity? I certainly do not "*now see*" that, and must leave myself open to further light. The answer to what Mr. Smith would be at, is in the previous paragraph.

Mr. Smith then very tenderly declines to analyze my Paper with greater "severity," but informs me that I have afforded him great amusement. I do not wonder that he declines to face the difficulties I have set before him, and to furnish the needful proof of his first principles. But so long as he refuses to do that, is it not a source of amusement to his critics to witness his faith and confidence in his creed?

Next, we have Mr. Smith's use of the equilateral. There is here the following:—Circle P has radius $= 2$; circle x has radius $= 4$; then, of course, 4π and 16π are the respective areas: and Mr. Smith tells us that $3\frac{1}{8}\ x = P^2$; that is, $3\frac{1}{8} \times 16\pi = (4\pi)^2$, and he asks: "how would this *be possible* if π were indeterminate? Of course it could'nt; but *in limine* I dispute the equation. Before you can form such an equation, you must first have found $\pi = 3\frac{1}{8}$. If you have already found this true, what further proof can this equation furnish?

Mr. Smith offers me another dilemma: 4π (P) $= \pi$(O D^2) and 4^2(P) $= 4$ O D^2, and I must find another value of π than $3\frac{1}{8}$ that will give these equations. The same answer as before is all that is needed here. Those equations cannot be formed unless you *previously assume* or prove $\pi = 3\frac{1}{8}$. How, then, can Mr. Smith be so absurd as put these things forward as a *proof* that $\pi = 3\frac{1}{8}$. The whole thing is a *card house*. It is not Mathematics, nor Geometry either, beyond the construction. These concentric circles have all a fixed ratio to one another. They are as the squares of radii, which radii are all fixed by the construction, but you can learn nothing whatever of π from them.

I am referred again to the equation, (O B^2 + B C^2 + O C^2) $= x$. Mr. Smith asks me to prove that this equation does not hold true. If it is true I have admitted $\pi = 3\frac{1}{8}$. Mr. Smith cuts the knot by asserting that it *is true*. He is surely quite aware, that to ask me to prove it *untrue* is not fair, according to the laws of controversy. He is bound to prove that it *is true*. My reply to him all along has been—You assume it, or assert it, without the shadow of a proof. It was this very point

substantially I pounced upon, in my criticism of the Buccleuch demonstration. It is the fundamental fallacy of all Mr. Smith's statements. I showed long ago that you can make π anything in this way. Let me divide the diameter in any ratio other than 7 : 1, and I have the very same right to assume and assert that the sum of the squares of the parts = the area of the circle, as Mr. Smith has to assume or assert, that the sum of the squares of his division = circle. For, his equating these two squares to the circle is the first assumption, on which, of course, the other, $OB^2 + BC^2 + OC^2$ is founded.

Mr. Smith next proceeds to prove $\frac{24}{25}$ (circumference) = perimeter of a regular hexagon. He starts with $\frac{\pi}{4} = 12 \cdot 5 \left(\frac{\pi}{50}\right)$; and claims admission for the fact that this = area of circle whose diameter = 1. Who ever disputed this ? Mr. Smith is at perfect liberty to start with it in any enquiry. He starts with it, and proceeds to prove that $\frac{24}{25}$ (circumference) = perimeter of a regular hexagon. The first step is the *assertion* that $\frac{25}{24}\, r \times$ twice the perimeter of a regular hexagon $= r \times 2\pi$; *i.e.*, $\frac{25}{24}(r)6 = 2\,\pi\, r$, and hence, $\frac{24}{25}$ (circumference) $= 6\,r$. Now, in this way, *anything* can be proved. You have merely to *assert* it, and call a man names if he does not admit your assertion. On the supposition that Mr. Smith is serious, and not seeking amusement, I cannot comprehend how he can call this Mathematics, or claim to have *proved* anything. If any one of all his friends or correspondents profess to be satisfied, so much the worse for the convert's claim *to be capable of judging* in a case like this.

At page 10, Mr. Smith admits what I have so long said—that he must supply a proof that π is determinate as a preliminary in his *mode* of finding π. He does not state the question accurately when he puts it thus : "Is it demonstrable that π must be a finite quantity ?" He makes me say, *No!* I do not say that π is *infinite;* but only that past efforts on the problem have shown that we can get nothing but a near approximation to the value. That there have been so many circle-squarers does not prove that it has appeared to the Mathematicians absurd and irrational

that π should be indeterminate. Judging from one or two specimens I have seen, it is not to their Mathematics that their efforts are due; and their fancied solutions are in the teeth of the first principles of that noble science.

Mr. Smith, however, declares that the determinate character of π may be proved in many ways. I wish he had *started* with one of them. On looking at this proof I find it is just the same as I have already disposed of in these notes. The equation, $\frac{\text{area of circle}}{\frac{1}{4}\pi}$ = circumscribing square, is true whatever π may be. All allow this; but what can we infer from it as to the nature of π? Mr. Smith's multiplying and dividing give him back the square of the radius. And I repeat that if $\pi = 3\sqrt{2}$, or, $\pi = 3\sqrt{3}$ or any other imaginable quantity, *surd* or *rational*, the equation $\frac{m}{\frac{1}{4}\pi}$, holds good. For, $\frac{m}{\frac{1}{4}\pi}$ is just $\frac{\pi r^2}{\frac{1}{4}\pi}$; or, in other words, $4\ (r^2)$. The multiplying, and dividing by the same number leave the multiplicand as it was, or rather multiplying, and then dividing by one fourth of that number quadruples multiplicand. You take the pea from one thimble, put it under another, and then restore it to No. 1. This proves nothing as to the character of π; and it leaves the great and fundamental question unsettled. I presume that the *many* other proofs are of a piece with this one.

I have thus noticed all in this Paper that bears on the first principles, and find not one solid point to stand on. I have dealt sufficiently with Mr. Smith's statements about the question *determinate* or *indeterminate*. I am not at all surprised that he cannot give good and sufficient *a priori* reasons for believing the latter alternative; but I am surprised at his faith in it. It is really an *irrational* faith, if he have no other grounds for it than he has yet stated to me. I appeal to Mr. Smith with all seriousness whether it is worthy of him to assail Mathematics with such a heresy as $\pi = 3\frac{1}{8}$ on grounds so utterly inadequate. It is not right to waste valuable time, and to call Mathematicians hard names in such a cause.

A first principle in Mathematics must always be plain and

near at hand. We have no difficulty with the axioms. The definitions of the various geometrical figures are simple. The whole science is built upon these, and all its truths are deducible by very simple methods; that is, simple to those whose intellect is of that order, and who care to give it the time and study needed to master them.

Now, I again assert that the question *determinate* or *indeterminate* as to π does not lie on the surface. That it is determinate is not involved in the definition of a circle; but if it be true, it must be *proved* to be one of the properties of the circle.

Now, this proof of Mr. Smith's is simply the following. Because the square of the diameter of a circle *may* be a determinate number, therefore, the ratio between the diameter and the circumference *must* be determinate. For in every case $\frac{m}{\frac{1}{4}\pi}$ is $= r^2$, or, $\frac{d^2}{4}$. It just occurs to me to say, that I should have denied another thing in the above *proof*. Mr. Smith says, that (n) is *indisputably* a *determinate* quantity. Now, suppose $r = 1 + \sqrt{2}$, will n be *determinate*? $r^2 = 1 + \sqrt{2} \times {}^{\cdot}2 = 3 \times (2\sqrt{2})$, is this determinate? Can we not construct a circle whose radius $= 1 + \sqrt{2}$? It is for this reason that I say in the above sentence, "*may be.*" But it may not be: for $2 + \sqrt{18}$, or $3 + \sqrt{32}$, or $5 + \sqrt{98}$, and many other quantities of an indeterminate character may be the radius; and hence: n may be indeterminate. What then comes of the above proof that should overcome the *prejudices* of Mathematicians? If this is Mr. Smith's strongest reason, it is more wonderful that he himself believes than that he cannot overcome, not the prejudices, but the common sense of Mathematicians.

BARKELEY HOUSE, SEAFORTH,
21st of April, 1868.

MY DEAR SIR,

The Paper of Mr. R——, upon which I commented somewhat freely in my Letter posted yesterday, is an extraordinary document: but his last Paper, which lies before be, is still more extraordinary.

Now, my dear Sir, you know as well as I do, that the area of an inscribed square to any circle=half the area of a circumscribing square: that, $\frac{\pi}{3}$ (perimeter of any regular hexagon) = circumference of a circumscribing circle: that, $\frac{\pi}{3}$ (area of any regular dodecagon) = area of a circumscribing circle: that, $\pi\ (r^2)$ = area in every circle. What would you think of me, if I were to say: Oh! but you must prove the value of π, before you can prove any of these things! Would you not think I had lost my common sense, if I ever had any? I am sure, if you thought me worthy of it, and attempted to reason with me, you would say:—With the first, π has nothing whatever to do; and as to the other three, they can be readily demonstrated, by means of any finite hypothetical value of π. Well, then, is it not self-evident, or axiomatic, that what is proveable by any *spurious* value of π, must be demonstrable by the *genuine article?* Now, can we prove anything arithmetically by an *indeterminate* value of π? Certainly not! Being unwell, you could not, as a matter of course, read Mr. R——'s last *elaborate* Paper, with your usual care; and probably forwarded it, through Miss S, even without perusal

Now, my dear Sir, I am sure you will hardly believe

it, but nevertheless 'tis true, that Mr. R——'s reasoning in this *extra*-extraordinary Paper, would make the four facts to which I have referred, *indemonstrable*, until we have first proved, "*a priori*," that π is a determinate arithmetical quantity ; that is to say, until we have proved by "*a priori and abstract reasoning*," that π cannot be an indeterminate quantity.

Now, let A B denote a straight line, and I admit that a straight line is the shortest between any two points, and let its length be represented by any finite arithmetical quantity, or, by the square root of any finite arithmetical quantity. Then : $4\ (A\ B^2)$ = area of a circumscribing square to a circle, of which A B is the radius. This can be demonstrated, *with the intervention of the circle*, by means of any finite arithmetical value of π, intermediate between 3 and 4.

For example: Let A B = 7, and, by hypothesis, let $\pi = 3{\cdot}16$. Then : Since $\pi\ (r^2)$ = area in every circle, it follows, that, $\pi\ (7^2) = 3{\cdot}16 \times 49 = 154{\cdot}84$ = area of the circle, on the hypothesis that $\pi = 3{\cdot}16$: therefore, $\frac{\text{area}}{\frac{1}{4}\pi} = \frac{154{\cdot}84}{{\cdot}79} = 4\ (A\ B^2)$; that is, $\frac{\text{area}}{\frac{1}{4}\pi} = 4\ (A\ B^2)$, and this equation = 196 = area of a circumscribing square to a circle of radius = 7. Hence, we may make π to be anything, so that it be determinate, interpose the circle, and by means of it prove the truth of the equation, $\frac{\text{area}}{\frac{1}{4}\pi} = 4\ (A\ B^2)$. Can we work out the calculation, and establish the equation, with an indeterminate value of π? Certainly not! This will be plain enough to any first class school-boy ; but, strange to say, a Mathematician cannot see it, so far is he carried beyond the regions

of common sense, by "*crammed erudition.*" Well, then, I gave Mr. R—— this proof that π cannot be indeterminate, in my Letter of the 11th inst., and connected it with the accompanying diagram; that is to say, I connected my proof with "*practical Geometry,*" which Mr. R—— *boldly asserts* can prove nothing.* That gentleman treats

* The following is the construction of the diagram:—Let A and B be two points. Join A B, and on A B describe the equilateral triangle O A B. From the angle O let fall a perpendicular, bisecting the angle and its opposite side A B at H. From the angle H draw a straight line parallel to O A, to meet and bisect O B at K. With O as centre and O K as interval describe the circle P. With O as centre and O H as interval describe the circle N. With O as centre and O B as interval describe the circle X. From the angle B draw a straight line B C, tangental to the circle X, making B C equal to $\frac{3}{4}$ (O B), and join O C. With O as centre and O C as interval describe the circle Y. From the angle C draw a straight line C F tangental to the circle Y, to meet O B produced at F, and with O as centre and O F as interval, describe the circle M. With C as centre and C O as interval describe an arc, to meet F C produced at D, and join D O. With O as centre and O D as interval, describe the circle Z. From F draw a straight line, F R tangental to the circle M, to meet O C produced at R; and with O as centre and O R as interval, describe the circle XZ. From R draw a straight line tangental to the circle X Z to meet O F produced at V. It is obvious that in this way we might extend the figure, *ad infinitum.* Produce C B to meet the circumference of the circle Y at P and join O P and P F. Then O B C, O B P, O C F, O P F, O F R, and O R V are similar right-angled triangles, and in all these triangles the sides that contain the right angle are in the ratio of 3 to 4, by construction. On O *e* describe the square O *e f g*, and on O *c* describe the square O *c v d.* Then, *e f*, *g f*, and C *v* are all tangental to the circle X. From D let fall a perpendicular to meet and terminate in the circumference of the circle Z at E. Produce O *f*, E O and *f* O to meet and terminate in the circumference of the circle Z, at the points N, L and T. From D draw a straight line at right angles to D F to meet and terminate in the circumference of the circle Z at M. Join D N, N M, and M L. Join *m n*, *n o*, *o p*, and *p m*, producing the square *m n o p*, which is an

my proof with silent contempt, and yet, oddly enough says:—"*Mr. Smith, however, observes that the determinate character of π, may be proved in many ways. I wish he had started with one of them.*" This quotation occurs in that part of his Paper written on the 16th inst., which professes to be a sort of running commentary on my Letter of the 11th, so that Mr. R—— can have no excuse for saying, that he had not seen what I offered as a proof by practical Geometry.

It would be a mere waste of time, as nothing would be gained by it, were I to comment upon Mr. R——'s last Paper, paragraph by paragraph. I shall therefore adopt another method of dealing with Mr. R——. I have no doubt that you, my dear Sir, have in the course of your long experience, more than once met with a precocious urchin, who fancied he knew everything and could do anything ; and by way of convincing him of his folly and presumption, have set him a task that you knew was beyond the reach of his intellectual capacity. Well, then, in dealing with a recognised Mathematician, it always comes to this at last, and it is unwillingly forced upon me, so to deal with Mr. R——.

Well, then, I give Mr. R—— the following theorem for solution, and I venture to tell you, my dear Sir, that you will find he cannot solve it. He must not say it is impossible, for if he will admit his incapacity to perform the task, I will shew him how it can be accomplished :—

inscribed square to the circle M. This square stands on the circle Y, and is exactly equal to the circle Y in superficial area. In the circle X inscribe the square *a b c d*. In the square *a b c d* inscribe the circle X Y ; and, in the circle X Y inscribe the square *a' b' c' d'*. To many of the properties of this geometrical figure, I had directed Mr. R——'s attention in previous Letters. (*See Diagram IX.*)

THEOREM.

Let the area of the circle P be represented by any finite arithmetical quantity. Find the arithmetical values of the sides of the right-angled triangle O R V, of which the side O R is the radius of the circle X Z; and find how many times the area of the circle P is contained in the area of the circle X Z.

If π were indeterminate, it would follow of necessity that the area of the circle P could not be contained an exact number of times in the area of the circle X Z; and in telling Mr. R—— that P is contained a finite number of times in X Z, I can assure that gentleman *I am not playing with him.*

I cannot say I have been amused with Mr. R——'s last Paper, but I can truly say it has pained me. In it he has resorted to wilful and glaring misrepresentation. This is a serious charge, and I must furnish you with the proofs. "*What cannot speak, cannot lie.*" Here then are the proofs.

The following paragraph is a literal quotation from page 2 of my letter of the 11th inst. :—

"Now, my dear Sir, mark the *dilemma* into which Mr. R—— has brought himself. He will not venture to dispute, that the area of a circumscribing square to any circle is found by dividing the area of the circle by $\frac{\pi}{4}$ whatever be the value of π. Now, if a denote the area of a circle, and b denote the area of a circumscribing square, then, $\frac{a}{\frac{1}{4}\pi} = b$, whatever be the value of π; and it follows of necessity, that if π be indeterminate, b must also be indeterminate. The conclusion to which Mr. R——'s

argument and reasoning would lead us, would be this. The diameter of an inscribed circle to any square, cannot be arithmetically expressible, either by a finite number or the square root of a finite number. Mr. R—— should have thought of this before he ventured to say of my conclusion, '*This is perfect nonsense.*'"

The following is Mr. R——'s commentary upon this paragraph:—

"On the 2nd page of this Letter Mr. Smith puts me in a dilemma! If, (a) be area of Circle, and (b) the side of circumscribing square, *that is, diameter:* then, $\frac{a}{\frac{1}{4}\pi} = b^2$, and it follows, that if π be indeterminate, b must be indeterminate also; and he actually asserts that my reasoning implies 'that the *diameter* of a circle inscribed in a square cannot be arithmetically expressible by any finite number, or by the square root of any finite number.' Now, *the side of a square* IS the diameter of the circle, and if *that side* be determinate, the diameter of the circle, being the same, must of course, be determinate also. There is no dilemma here—not the shadow of a horn. I would simply say that, in such a case (a) the area is indeterminate, but being divided by π, the quotient $\frac{a}{\frac{1}{4}\pi}$ will of course give you back b, after the little shuffle of the cards, $a = \frac{1}{4}\pi b^2$ and $\frac{a}{\frac{1}{4}\pi} = \frac{\frac{1}{4}\pi b^2}{\frac{1}{4}\pi} = b^2$. This is surely plain enough, without any more figuring."

Now, you will observe, my dear Sir, that Mr. R——, first, glaringly and wilfully misrepresents my *premiss*, by substituting *side* of circumscribing square, for *area* of circumscribing square; and then applies my reasoning to his *own premiss*. Could anything be more monstrous?

Again: The following paragraph is another literal quotation from page 2 of my Letter of the 11th instant:—

"Well, then, Mr. R—— admits the following fact: 'No other proof than the simple statement of the ratios, is needed to show that $3\frac{1}{8}\ x = z$, when the radius of $x = 4$, and the radius of $z = 5\ (\sqrt{2})$; and he then puts the question: But how does this upset the fact that π is indeterminate? Surely Mr. R—— will now see, that in *assuming* it to be a *fact* that π is indeterminate, he might as well assume that the area of a square is indeterminate! Could absurdity go further? Well, then, it is not necessary to analize Mr. R——'s last paper with greater severity; but, I may assure him, in some respects, it has greatly amused me."

The following is Mr. R——'s commentary upon this paragraph:—

"The next paragraph on page 2 is not intelligible in its present form. $Z = 3\frac{1}{8}\ x$: and I asked 'how this upsets the fact that π is indeterminate?' Mr. Smith replies that in *assuming* it to be a *fact* that π is indeterminate, I might as well assume that the area of a square is indeterminate. Could absurdity go further? What absurdity? 'I certainly do not *now see*' that; and must leave myself open to further light. The answer to what Mr. Smith would be at is in the previous paragraph."

Now, what I say is this. If (a) denote the area of a circle, and (b) the area of a circumscribing square, and π be indeterminate, it follows, that $\frac{a}{\frac{1}{4}\pi}$ which $= b$, must make b also indeterminate, and would make the side of the circumscribing square arithmetically inexpressible, either by a finite number, or the square root of a finite number; and even Mr. R—— cannot fail to perceive that the side of a

square must be expressible either by one or the other. But, let $(a) = 5$, and let π be determinate, say $3{\cdot}1416$. Then: $\frac{a}{\frac{1}{4}\pi} = b$, that is, $\frac{a}{{\cdot}7854} = b$, and this would make b indeterminate; for, $\frac{5}{{\cdot}7854}$ is an indeterminable decimal, and consequently the side of the square and diameter of inscribed circle, cannot be expressed either by a finite quantity, or the square root of a finite quantity. Then: would it not be glaringly absurd to say, that if $\frac{a}{\frac{1}{4}\pi} = b$, and π be indeterminate, that b is determinate? Now, Mr. R—— again attempts to get out of the difficulty by falsely stating my premiss, and applying my reasoning to a premiss of his own, when he says :—"*The answer to what Mr. Smith would be at is in the previons paragraph.*"

Now, my dear Sir, what am I to do? What can I do with an opponent, when that opponent first catches at a statement, perverts its meaning, and distorts my reasoning; and yet, finding that by such means he fails to get up a valid argument, then resorts to glaring and wilful misrepresentation of facts and plain statements! Controversy, with such an opponent, is hopeless, and would be interminable. Well then, instead of further controversy, let Mr. R—— solve my theorem!

There are three things involved in, and essential to a Circle, which is the most perfect figure in nature, viz.: circumference, diameter, and area. "*A circle is a plain figure bounded by one line called the citcumference or periphery, to which all straight lines drawn from a certain point within the figure, called the centre of the circle are equal.*" "*A diameter of a circle is a straight line drawn through the centre, and terminating on both sides with*

the circumference." The circumference of a circle encloses a space, called the area, as to which theoretical Geometry proves nothing, teaches nothing. It is practical Geometry combined with Mathematics, that teaches us that semi-circumference × radius = circumference × semi-radius, and that this equation = area in every circle; but this proves nothing, teaches nothing, as to the true value of the symbol π; for, one finite value of the symbol π is just as good as another to prove the equation. Hence: $\pi (r^2)$ = semi-circumference × radius, or circumference × semi-radius, whatever be the arithmetical value of π. Well then, for the purpose of ascertaining the ratio of diameter to circumference in a circle, or in other words, the number of times the diameter of a circle is contained in the circumference, no such symbol as π is required. But, Mathematicians having failed to discover the number of times the diameter of a circle is contained in the circumference, and consequently, not being able to give exact arithmetical expression to it, have adopted the symbol π to represent it.

Now, Mr. R—— is of opinion that it is not *irrational* to suppose that π is indeterminate—and in this all recognised Mathematicians will agree with him, or at any rate, they will say so—but is it not both *irrational* and *absurd* to suppose that if one of the three essentials to a circle be given, that is, circumference, diameter, or area, we cannot find the other two? Mr. R—— might as well tell me, that if the area of a square be given, it is not *irrational* and *absurd* to suppose that we cannot find the values of the side and perimeter of the square, and the diameter of an inscribed circle. I maintain that, if the area of a circle be given, it is both *irrational* and *absurd* to suppose that we cannot

find, with arithmetical exactness, the diameter and circumference: and, not more *irrational* and *absurd* than *false*. Well, then, let Mr. R—— solve my theorem and he will be of the same opinion. He will solve my theorem *if he can*, and be a scientific truth-seeker: and, *if he cannot* solve my theorem, and be a candid Mathematician, he will frankly admit his inability to do it: and then, I will—as I have already said—shew him how to accomplish the task. This, my dear Sir, may appear egotistical, but with such an opponent as Mr. R——, what am I to do? What can I do?

In the facts I have brought under your notice, I can assure you, I am very far from exhausting all the properties of the geometrical figure represented by the enclosed diagram. In my Letter of the 6th Instant, I directed Mr. R——'s attention to the fact, that T O E a quadrant of the circle Z, is divided into three angles, T O P, P O F and F O E; and since T O E is a right angle, it follows, that the three angles T O P, P O F, and F O E are together equal to to a right angle. I requested Mr. R—— to be good enough to tell us the values of these angles, expressed in degrees, and the arithmetical values of their natural sines. This request he has treated with silent contempt. Mr. R—— *asserts* that he knows everything I can offer him in Geometry, and *assumes* that he knows far more than I do in Mathematics. Why, then, should he be afraid to grapple with the truths I bring under his notice? In the Paper that lies before me, Mr. R—— observes:—" *I am referred again to the equation* $(O B^2 + B C^2 + O C^2) =$ X. (This is a fact, and I defy Mr. R—— to disprove it). Mr. Smith *asks me to prove that this equation does not hold true. If it is true, I have admitted* $\pi = 3\frac{1}{8}$. Mr.

Smith *cuts the knot by asserting that it is true.* (This assertion is not true, for I have proved the fact by many equations). "*He is surely qaite aware that to ask me to prove it untrue is not fair according to the laws of controversy. He is bound to prove that it is true.*" When I prove that ($O B^2 + B C^2 + O C^2$) $= X$, by equation upon equation, I am met with the *assertion* "*I dispute the equation,*" without a shadow of proof. If I try to get at Mr. R—— by giving him a problem or a theorem to solve, he assumes this to be unfair, and declines to do either. Then, what am I to do with such an opponent? What can I do? Would it not be useles and absurd to continue the discussion?

In closing the controversy with Mr. R——, I must not omit to thank you very sincerely for the interest you have taken in it, and the promptitude with which—as the medium between us—you have forwarded my communications. I hoped for a happier result, but it cannot be helped: I must bide my time. "*Magna est veritas, et prevalebit.*"

Mrs. Smith joins me in kind regards to yourself, Mrs. and Miss S——, and we hope you are recovering from your attack of inflammation.

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

BARKELEY HOUSE, SEAFORTH,

23rd April, 1868.

MY DEAR SIR,

I can assure you it has caused me much pain to write the enclosed Letter, and I am afraid the reading of it will cause you pain. I cannot tell you how much I regret this, when I think of your kindness, and the hope you had that you had brought me into communication with a Mathematician, at whose hands I should get fair play. How sadly am I deceived!

In the enclosed Letter I give a theorem for solution. The solution is only possible by means of the true value of π; but the true value of π being known, the solution is simple enough. You may depend upon it, that Mr. R—— will not attempt to solve it, but stand upon his right, and deny my right to ask him to prove anything.

In a day or two I will send you the solution of the theorem, and I think it will convince you, that I have not had, and could not hope to get, fair play at the hands of Mr. R——, and that the only course left me is to drop the discussion.

If I have given you pain, I pray you pardon me; but as truth admits of no compromise, I could not, in the defence of it, do differently from what I have done.

Believe me, my dear Sir,

Faithfully yours,

JAMES SMITH.

J—— S——, Esq.

BARKELEY HOUSE, SEAFORTH,
24th April, 1868.

MY DEAR SIR,

I will now fulfil the promise made in my note of yesterday, which accompanied my Letter dated 21st inst., and intended for Mr. R——. In that Letter, I gave the following theorem, and I shall now proceed to give you the solution :—

THEOREM.

Let the area of the circle P be represented by any finite arithmetical quantity. Find the arithmetical values of the sides of the right-angled triangle O R V, of which the side O R is the radius of the circle X Z ; and, find the number of times the area of the circle P is contained in the area of circle X Z.

Let the area of the circle P = 60. Then : 8 circumferences of a circle = 25 diameters; therefore, $\frac{25}{8} = 3{\cdot}125$ is the number of times the diameter of a circle is contained in the circumference: $\frac{3{\cdot}125}{4} = {\cdot}78125$ is the area of a circle, when the diameter = 1. Hence: $25\,({\cdot}78125^2) = 25 \times {\cdot}6103515625 = 15{\cdot}2587890625$; and, $15{\cdot}2587890625$ is the number of times the area of the circle P is contained in the area of the circle X Z ; therefore, $60\,(15{\cdot}2587890625) = 915{\cdot}52734375$ = area of the circle XZ. $\sqrt{\left(\frac{915{\cdot}52734375}{3{\cdot}125}\right)} = \sqrt{292{\cdot}96875}$ = O R the radius of the circle X Z. $\frac{3}{4}$ (O R) = $\frac{3}{4}\,(\sqrt{292{\cdot}96875}) = \sqrt{164{\cdot}794921875}$ = the side R V in the right-angled triangle O R V. But the sides O R and R V contain the right angle, therefore, OR^2

$+ RV^2 = 292{\cdot}96875 + 164{\cdot}794921875 = 457{\cdot}763671875 = OV^2$; therefore, $\sqrt{457{\cdot}763671875} = OV$ the hypothenuse of the triangle O R V.

Proof: Because the radius of the circle X = the diameter of the circle P, by construction; the area of the circle = X = 4 times the area of the circle P; therefore, 4 (area of circle P) $= 4 \times 60 = 240 =$ area of the circle X; and, because the square of the radius multiplied by the number of times the diameter is contained in the circumference = area in every circle, it follows of necessity, that the square root of area divided by the number of times the diameter is contained in the circumference = radius in every circle; therefore, $\sqrt{\frac{240}{3{\cdot}125}} = \sqrt{76{\cdot}8} = OB$, a radius of the circle X. But, the right-angled triangles O B C, O C F, O F R, and O R V are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 3 to 4, by construction. Hence: $\frac{5}{4}(OB) = \frac{5}{4}(\sqrt{76{\cdot}8}) = \sqrt{120} = OC$ the radius of the circle Y. $\frac{5}{4}(OC) = \frac{5}{4}(\sqrt{120}) = \sqrt{187{\cdot}5} = OF$ the radius of the circle M. And, $\frac{5}{4}(OF) = \frac{5}{4}(\sqrt{187{\cdot}5}) = \sqrt{292{\cdot}96875} = OR$ the radius of the circle X Z. $\frac{3}{4}(OR) = \frac{3}{4}(\sqrt{292{\cdot}96875}) = \sqrt{164{\cdot}794921875} =$ the side R V, in the triangle O R V. And, $\frac{5}{4}(OR) = \frac{5}{4}(\sqrt{292{\cdot}96875}) = \sqrt{457{\cdot}763671875} = OV$, the hypothenuse of the right-angled triangle O R V; therefore, $OR^2 + RV^2 + OV^2, = (292{\cdot}96875 + 164{\cdot}794921875 + 457{\cdot}763671875) = 915{\cdot}52734375 =$ the sum of the areas of squares about the triangle ORV. But, OR^2 multiplied by the number of times the diameter of a circle is contained in the circumference = area of the circle

XZ; therefore, $OR^2 \times 3{\cdot}125 = 292{\cdot}96875 \times 3{\cdot}125 = 915{\cdot}52734375$ = area of the circle XZ; therefore, area of the circle $XZ = 15{\cdot}2587890625$ times the area of the circle P; therefore, $8\,(15{\cdot}2587890625) = 122{\cdot}0703125$ = area of the circle M, when the diameter of the circle $P = 4$.

In this proof the symbol "π *does not appear, need not appear*," but the proof itself demonstrates beyond the possibility of dispute or cavil, that the arithmetical value of the symbol π, which denotes the number of times the diameter of a circle is contained in the circumference = $3{\cdot}125$.

Hence: $(OR^2 + RV^2 + OV^2) = 3\frac{1}{8}\,(OR^2)$, and this equation = area of the circle XZ; and since the property of one circle is the property of all circles, it follows, that the area of a circle = the sum of the areas of squares about a right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4, and the longer of those sides the radius of the circle.

Well then, I have now, my dear Sir, furnished you with the solution of the theorem given in my Letter of the 21st. This of course is not intended for Mr. R—— at present, but you will have it in your possession, and it will enable you to convince yourself, whether I have had a fair opponent in Mr. R——.

Again thanking you for the trouble you have been at, and the interest you have taken in my scientific labours, and hoping you are rapidly recovering from your attack, with kind regards,

Believe me, my dear Sir,

Very faithfully yours,

J—— S——, Esq. JAMES SMITH.

Mr. R——'s PAPER, *April 23rd*, 1868.

I have carefully read Mr. Smith's Letter of the 14th, containing, chiefly, remarks on some notes of mine, on his Paper of 15th February.

I had understood that Mr. Smith revealed the whole process by which he convinced himself that $\pi = 3\frac{1}{8}$; and my impression was natural and warranted. I had requested that gentleman "to relate the whole steps of the process by which he discovered that number to be the ratio." And in reply, in his Letter 15th February, he wrote: "I have no wish to conceal, or, for a moment hesitate to reveal, the process of reasoning by which I arrive at that conclusion." All along I understood Mr. Smith to mean what, I am sure, every person comparing my request with his reply, would *understand* him to mean—that he "related the whole steps of the process." Now, at length, Mr. Smith emphatically assures us that, he has only "revealed the *first steps* in his process."

Now I might complain of this. But I rather take up this new confession, and of it affirm, that, if we have the *first steps* in the assumptions which I criticised, and of which I have hitherto, in vain, requested satisfying evidence, the steps that follow can be of no use. The first steps are the first principle—the foundation—and if these be not sound, truth is unattainable by that process.

Mr. Smith has not yet faced *broadly* and manfully my request for evidence on two points. First: Distinct *a priori* evidence that the ratio must be expressible in a terminate decimal. He lays down as if it were axiomatic: (first) that this ratio must be a determinate arithmetical quantity, and second, that this quantity must be expressible in decimals that terminate.

Now, these are two conditions which neither of them lie on the surface. Mr. Smith might condescend to the weakness and blindness of an exotic like me, and make plain why these

two conditions fence round this sacred and ethereal π. $3\frac{1}{7}$ is, in all conscience, a *determinate* number. If the diameter (d) is given, can you not draw a line $= 3\frac{1}{7}\ d$? My Geometry teaches me to do this: to find a line absolutely $=$ to $3\frac{1}{7}\ d$! It is therefore a *determinate* ratio; but $\frac{1}{7}$ cannot be expressed in decimals, because of course 7 and 10 are incommensurable.

Why then this *second condition*, that the fractional part of the ratio must be expressible *decimally*? Mr. Smith, by imposing these *two conditions* as his π, makes his difficulties rather considerable. It is obvious to me that these conditions are purely arbitrary; that there is nothing that imposes them in the nature of the case; and that Mr. Smith by a sort of *ipse dixit* erects them as a kind of defence of his π.

There is nothing irrational *a priori* in the admission that π *may* be indeterminate; the irrationality is in capriciously or arbitrarily fixing its nature before we FIND IT. Let us see it, and then shall we know whether terminate or otherwise. This latter is the *rational* mode of the Mathematicians. He sweeps the ground clear of all conditions, and considers alone the definition of a circle, and the properties of that figure as they lie in the definition. And he enquires "By what mode can I find the ratio between the diameter and the circumference?" He does not lay down *a priori*, as self-evident, that this ratio MUST be a number expressible in DECIMALS. I consider that would manifestly be irrational, because there is nothing on the face of the circle that requires it or warrants it. It *may be true*, but he must *discover* it, and prove it; he must not assume it, and assert it, and claim that it shall be admitted at the outset, as Mr. Smith does. Here is the fatal hole in the coat: the lack that makes it no coat at all, not the shadow of a coat. Show me the coat, before I look for the holes. I cannot move one step till you satisfy me that the first step is in the right direction. We are going in search of π in this circle; he lies here; but how shall we find him? Oh! says Mr. Smith, he is a determinate quantity! How do you know? How *can* you know until you *find* him? Oh! says Mr. Smith, see, he is $3\frac{1}{8}$; is he *not* determinate?

Rather cool this! Then you have *found* him *already*; what need to go in search of him, if you have found him before we set out on the search? This is exactly Mr. Smith's answer to my demand. He *finds* this π by some process which he keeps sacredly concealed, and wants us to have faith in his skill and wisdom!! Sir Isaac Newton, himself, did not ask so much of mankind.

My *second request* has been for evidence to show what other limits exist than 3 and 4.

In connection with this, I have to say, that while 3 and 4 are demonstrably the limits of π, I do not know any other limits, *a priori*, or by any existing proof. I have never admitted Mr. Smith's claim to fix it between $3\frac{1}{8}$ and $3\frac{1}{7}$; and his reasons for doing so are somewhat strange, namely, that the orthodox π is less than $3\frac{1}{7}$. I did not think that IT could come into court at all. Let Mr. Smith be assured, that I will admit any limits that he can prove as clearly as 3 and 4 are proved; but I have not yet seen or heard any proof that the ratio cannot be $3\frac{1}{7}$, but *must* be *less* except the same proof which also shows it to be *more* than $3\frac{1}{8}$!

On page 19 there is a sentence in point. On the general question, Mr. Smith says that about a circle we may describe a square, and about that square a circle, and so on *ad infinitum*. This of course is plain; and it is plain also that each succeeding circle and square would be double of the previous ones respectively. They would be represented by a series of which 2 would be the common ratio:

$$\text{Circles } \pi r^2, 2\pi r^2, 4\pi r^2, 8\pi r^2, 16\pi r^2.$$
$$\text{Squares } 4r^2, \; 8r^2, \; 16r^2, 32r^2, \; 64r^2.$$

Then says Mr. Smith: "But if π were indeterminate, the areas of circles would be inderterminate also, while the areas of the squares would be determinate, and no definite ratio could exist between the areas of the circles and the areas of the squares, which is not only irrational but absurd. We know that the area of every circle and of every square is the double of that which precedes it; then, how in the name of common sense can the

areas be indeterminate?" And then to sum up it is asserted that the determinate character of π is as obvious as that the *centre of* a circle *is its centre.*

And then comes one of those clenching truisms by which Mr. Smith seems to think he annihilates all opponents. "Mr. R—— will not venture to tell me that we must first find the value of π before we can prove that the area of the circumscribing square = twice the area of the inscribed square." Of course, there is no fear of that. But I say that you cannot infer that the circles are expressibly determinate in terms of the diameter because the *squares* are determinate. Does Mr. Smith observe, that the side of each alternate square $8r^2$, $32r^2$, $128r^2$, is inexpressible determinately by any decimals. Yet $\sqrt{2}$ is a determinate quantity. My geometry teaches me to draw the line represented by it.

In one word, on these two points, Mr. Smith most manifestly fails to assign a single proof of these two first principles, or as he now says, "first steps in his process," those that π is *arithmetically* determinate, or rather expressible in decimals, for that is the condition—one of the fetters Mr. Smith has imposed on himself: and (second) that there are limits within 3 and 4 which *a priori* can be shown to confine π.

If Mr. Smith be himself convinced by his own reasons, I may venture to say that no other person capable of understanding, and who does understand the problem, will be convinced. I shall assume this, aye, and until some other man comes forward and declares himself convinced, and shows how he obtained satisfaction. If there is anything that a mathematical training begets in a human mind, it is a clear eye to first principles, and to the strictly legitimate deduction of other truths from them. I frankly confess that Mr. Smith's faith in $3\frac{1}{8}$ is most marvellous to me. In fact my chief effort all along has been to get his stand point, so to speak, and find out the lights—the delusive lights—in which he looks at this matter, and see thus *how* he came to a conclusion which he cannot establish, and which no mathematician has looked at for a moment,

during these 6 or 7 years of Mr. Smith's controversy with them. I have not yet been able to fathom how one with even a slight knowledge of geometry could offer for instance that document which was addressed to the Duke of Buccleuch as a sound demonstration.

In another sentence, Mr. Smith asks me to reflect on the absurdity of supposing that every successive circle and square will be double of its predecessor, and yet the area of the circles determinate arithmetical quantities. If this is not self-evident, I have only to put a finite value on area Y, and that will make the diameter of every circle and the area of every square, indeterminate, in working the calculations with the determinate π.

If Mr. Smith look at the series πr^2, $2 \pi r^2$, $4 \pi r^2$, $8 \pi r^2$, &c., he will see that all you can say about the numbers of it is just that they have a fixed ratio one to another. If even you put a value on r, the *diameters* of the alternate circles are not expressible arithmetically. What does Mr. Smith infer from $2 r \sqrt{2}$, $4 r \sqrt{2}$, $8 r \sqrt{2}$, which are respectively the diameters of the alternate circles and the sides of their circumscribing squares. These cause me no difficulty. And, in one word, looking at that series of circles and squares, you see the ratio between them ; but you can infer nothing from that as to the ratio between the diameters and circumferences. The arithmetical value of π need not appear ; it is the same in all possible circles, *ad infinitum.*

On page 17 there is a sentence which has frequently appeared in Mr. Smith's Letters. To find π, you must connect it with some other figures in the diagram, and produce results to which every other but the true value of π is utterly incompetent. And then he states that $3\frac{1}{8}$ squares of the radius of a series of circles = square of sides of certain triangles, whose sides are as 3, 4, 5. Now the sides being so related, we have of course $3\frac{1}{8}$ (16) $= 3^2 + 4^2 + 5^2$: so that in triangles thus formed, $3\frac{1}{8} \times$ square of the middle side = sum of the squares of the rest. $3\frac{1}{8}$ times $(4 a)^2 = a^2 (3^2 + 4^2 + 5^2) = 50 a^2$. This is true, always true.

As you vary a, you get a series of triangles of the kind in Mr. Smith's diagram, and $3\frac{1}{8}$ will be the quotient of $\frac{(3a)^2 + (4a)^2 + (5a)^2}{(4a)^2}$. This is of course plain. But that is all. You simply show that $3\frac{1}{8}$ is the quotient of $\frac{50}{16}$, for that is what it comes to; nothing but $3\frac{1}{8}$ can be *the* quotient, whatever a be from $\frac{1}{9999}$ to 9999: for in every case a^2 is common to both sides of the equation. But how does it follow from this that $3\frac{1}{8}$ is π? This is to me the astounding *fact*, that Mr. Smith should impose upon himself with such a piece of *reasoning*. Look into it and it leads you back to the assumption that when diameter is divided as 7 to 1, the sum of the squares = area of the circle, which Mr. Smith tried to prove to the Duke of Buccleuch. It is perfectly preposterous to ring the changes thus, and make up for lack of proof, by loudness of assertion. Mathematics are calm and clear—clear and calm as the blue heavens, without a cloud; nothing is admissible but pure reasoning, every step of which the attentive mind follows and receives with resistless conviction. The fact that all Mr. Smith's correspondents have pronounced against him should shake his confidence, and lead him to look without prejudice, or *feeling*, at his principles and process.

One other sentence, page 12. "If with $\pi = 3\frac{1}{8}$, a square = to a given circle can be constructed and exhibited—and that such square exists is admitted by geometers—what else can π be but $3\frac{1}{8}$? If diameter = 8, will Mr. R—— tell me that $3\frac{1}{8}\, r^2$ is not $= 7^2 + 1^2$?" The last sentence needs no denial. What I object to is not $3\frac{1}{8}\, r^2 = 7^2 + 1^2$, but $3\frac{1}{8}\, r^2 = 7^2 + 1^2 = (\pi\, r^2)$. Enough has been said on this above. What I wish to notice in this sentence is the ambiguity that pervades it, and misleads Mr. Smith. He says "if with $\pi = 3\frac{1}{8}$ a square = to a given circle can be constructed—and that such square exists is admitted by geometers—what square? What do geometers admit? Geometers do not question that a square equal to a given circle exists; what they question is the power to exhibit it with absolute accuracy. They can come *infinitely* near it; but there is always an infinitely small lack. The IF at the

beginning of the sentence is the great barrier. Here it is *dubitative*, conditional; not concessive; it is really IF not SINCE.

In page 4, Mr. Smith complains of my "*playing* with one part of a quotation and treating the remainder of it with contempt." "Do not the two conclusions stand or fall together?" That is just why it is not necessary to waste time over both. If you bring down one you bring down both. That $\frac{24}{25}$ circumference $=3$ (diameter), of course depends on $\pi=\frac{25}{8}$. If I question $\pi=\frac{25}{8}$ what use of going further. When Mr. Smith offers an *independent* proof that $\frac{24}{25}$ circumference $=$ perimeter of a regular hexagon, I shall gladly look at it. I have asked this too often without result, to cherish any hope of seeing it.

The alleged proof by continued proportion seems too ridiculous for me to take any more notice of it.

MR. SMITH *to* MR. J—— S——.

Q—— A——, DUMFRIESSHIRE,

27th April, 1868.

MY DEAR SIR,

Yesterday Miss S—— kindly handed me Mr. R——'s Paper, in reply to my Letter of the 14th inst. It has had my careful attention, and I shall reply to it, although, as you know, it was my intention to drop the controversy.

In a mathematical—or, indeed, in any discussion, it is not only essential that the disputants should be agreed on first principles, but also that each should understand

the terms employed by the other in the course of it, in the same sense. It appears to me, from Mr. R——'s last Paper, that there is one point upon which we have not hitherto understood each other, but upon which we are after all agreed, and discovering this fact, gives me some hope that we may yet arrive at "*a happy state of concord*" on the value of π, and the true ratio of diameter to circumference in a circle, and I am thus induced to renew the correspondence.

On page 3 of Mr. R——'s last Paper, he observes:—"*And then comes one of those clenching truisms by which* Mr. Smith *seems to think he annihilates all opponents.*" "Mr. R—— will not venture to tell me that we must first find the value of π before we can prove that the area of the circumscribing square = twice the area of the inscribed square. Of course there is no fear of that. But I say that you cannot infer that the circles are expressibly determinate in terms of the diameter, because the *squares* are determinate. Does Mr. Smith observe that the *side* of each alternate square $8\sqrt{r}$, $32\sqrt{r}$, $128\sqrt{r}$ is inexpressible determinately by any decimals? Yet the $\sqrt{2}$ is a determinate quantity. My geometry teaches me to draw the line represented by it."

It has often been asked of me:—Is not $\sqrt{2}$ incommensurable? An answer in the affirmative must, of course, be the reply. From this, opponents have drawn the conclusion that "*if we make the diameter of a circle finite, away goes circumference and area into decimals without end; and, if we make the circumference of a circle finite, away goes diameter and area into decimals without end.*" I quote from memory, but I believe I quote literally, the language of a correspondent, who was a first class

wrangler of his year. How, then, was I to suppose Mr. R—— was of a different opinion? He has never said anything to lead me to think so, until I received his last Paper, which lies before me.

Mr. R—— now admits that an arithmetical expression may be incommensurable, and yet be a determinate quantity; that is to say, he now admits that the $\sqrt{2}$ (which, so far as my experience goes, has always been a puzzle to Mathematicians) although an incommensurable, is nevertheless a *determinate quantity.* Well, then, on this point Mr. R—— and I are agreed. We are also agreed that the circumferences of circles are to each other as their radii. Now, let us enquire what follows of necessity from these data, upon which Mr. R—— and I are happily agreed.

Well, then, referring you to the diagram enclosed in my Letter of the 14th inst. *(See Diagram IX.)*, O K, O c', O H, O B, O C, O p, O E, and O R, are the radii of the respective circles, P, X Y, N, X, Y, M, Z, and X Z; and, if the letters which represent the circles denote their areas, $\pi(OK^2) = P$; $\pi(Oc'^2) = XY$; $\pi(OH^2) = N$; $\pi(OB^2) = X$; $\pi(OC^2) = Y$; $\pi(Op^2) = M$; $\pi(OE^2) = Z$; and, $\pi(OR^2) = XZ$.

Let the letters which represent the circles also denote their circumferences. Then: $2\pi(OK) = P$; $2\pi(OB) = X$; $2\pi(OC) = Y$; $2\pi(OF) = M$; and, $2\pi(OR) = XZ$; and the triangles O B C, O C F, O F R, and O R V, are similar right-angled triangles, and have the sides that contain the right angle in the ratio of 3 to 4, by construction. Hence: $3\frac{1}{8}(OB^2) = (OB^2 + BC^2 + OC^2)$; $3\frac{1}{8}(OC^2) = (OC^2 + CF^2 + OF^2)$; $3\frac{1}{8}(OF^2) = OF^2 + FR^2 +$

$O R^2$); and, $3\frac{1}{8}$ ($O R^2$) = ($O R^2 + R V^2 + O V^2$). But, 2 ($3\frac{1}{8}$) = 6·25. Hence: 6·25 (OK) = P; 6·25 (OB) = X; 6·25 (OC) = Y; 6·25 (OF) = M; and, 6·25 (OR) = XZ.

Now, let O K = 2. Then: O B = 4; O C = 5; O F = 6·25; O R = 7·8125; and, O V = 9·765625; therefore, O V = $(3\frac{1}{8})^2$. But, the equilateral triangle O A B is the generating figure of the diagram; and K H = O K, by construction; therefore, K H = radius of the circle P. Now, the equilateral triangle O A B is divided into two similar and equal right-angled triangles, since the straight line OH is drawn from the angle O perpendicular to its opposite side A B; therefore, $O B^2 - B H^2 = O B^2 - K H^2$, and this equation = $(4^2 - 2^2)$ = (16 — 4) = 12 = $O H^2$; therefore, O H = $\sqrt{12}$, and O H is the radius of the circle N.

Now, 6·25 (O K) = 6·25 (K H), and, by hypothesis, O K = 2; therefore, 6·25 × 2 = 12·5 = P, when the letters that represent the circles, also denote their circumferences; and, 6·25 (O H) = 6·25 ($\sqrt{12}$) = $\sqrt{(6{\cdot}25^2 \times 12)}$ = $\sqrt{(39{\cdot}0625 \times 12)}$ = $\sqrt{468{\cdot}75}$ = N. Hence: N is *determinate*, although incommensurable. Now, it follows of necessity, that 6·25(OH) : circumference of the circle N : : 6·25 (O B) : circumference of the circle X. But the circumference of the circle N = $\sqrt{468{\cdot}75}$; therefore, O H : N : : O B : X, that is, $\sqrt{12} : \sqrt{468{\cdot}75} :: 4 : \sqrt{625}$; therefore, $\sqrt{625}$ = 25 = X; therefore, the circumference of the circle X, is not only *determinate*, but commensurable.

I shall not trouble Mr. R—— with any more "*figuring*," as he may readily verify the following facts. The ratio of area to area in all the circles, is *determinate* and commensurable, and, the area of the circle P is contained

15·2587890625 times in the area of the circle X Z; and Mr. R—— will find that this fact is demonstrable, by whatever finite hypothetical arithmetical value of π he may work out the calculations. And having now admitted that the $\sqrt{2}$ is *determinate*, although incommensurable, Mr. R—— will also find that the ratio of radius to circumference in all the circles is *determinate*, although the circumferences of three of these circles are incommensurable, while the circumferences of the other five, are not only *determinate* but commensurable. Then I would again ask Mr. R——: How can the value of π be an indeterminate arithmetical quantity?

Believe me, my dear Sir,

Yours very truly,

JAMES SMITH.

J—— S——, Esq.

MR. JAMES SMITH *to* MR. S——.

CROWN HOTEL, BOWNESS,
WINDERMERE, *2nd May*, 1868.

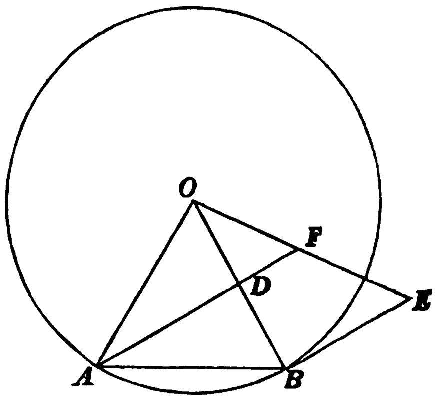
MY DEAR SIR,

Mrs. Smith found herself so much better for her trip to Scotland, that she was unwilling to return home direct, and so we found our way here on Thursday evening.

This morning is not propitious for an out-door ramble, and it has occurred to me by way of killing time, to drop you a short Letter, in which I think I can give a problem to Mr. R—— for solution, that will convince that gentle-

man of the absurdity of his assertion that "*Practical Geometry can prove nothing.*"

On the straight line AB describe the equilateral triangle O A B, and with O as centre and OB as interval, describe a circle. From the angle A draw a straight line perpendicular to its opposite side O B, bisecting O B at D. From the angle B draw a straight line BE tangental to the circle,



so that when O E are joined, producing a right-angled triangle O B E, A D produced shall bisect O E the hypothenuse of the right-angled triangle O B E at F.

It is obvious that there can be but one ratio between the sides O B and B E in the right angled triangle O B E. Well, then, let Mr. R—— find the ratios of side to side, in the triangle O B E.

Now, Mr. R—— has said that I can offer him nothing in Geometry that he cannot understand, and this being so, he can have no difficulty in solving this problem. I am sure you will be of opinion, that there is nothing unreasonable in my making this request, and if Mr. R—— solve the problem, it cannot fail—not only to convince him of the absurdity of his assertion, that "*Practical Geometry can prove nothing,*" but —— to teach him something that must necessarily lead to our arriving at

"*a happy state of concord,*" with reference to the question upon which we have been so long at issue.

Hoping you continue to make rapid progress towards convalescence, and with kind regards to yourself, Mrs. and Miss S——, in which Mrs. S—— joins me,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

MR. JAMES SMITH *to* MR. S——.

CROWN HOTEL, BOWNESS,
WINDERMERE, *4th May*, 1868.

MY DEAR SIR,

From the omission of a few words in my Letter of Saturday, my meaning would not be intelligible to Mr. R—— in the form I put it; and he might even imagine that I was setting a trap, and not acting candidly and fairly towards him. Now, I have no other wish, my dear Sir, but to convince Mr. R——, by fair argument, and no desire or intention to "*play*" with him; hence my writing to day; and since he admits I can offer him nothing in Geometry that he cannot understand, I will make my meaning unmistakeably intelligible to Mr. R——.

In the enclosed diagram A *(see Diagram X.)*, let OAB be an equilateral triangle. From the angle A draw a straight line perpendicular to its opposite

side O B, bisecting O B at D. With O as centre and O B as interval, describe the circle. From the angle B draw a straight line B E tangental to the circle. With O as centre and 2 (O B) as interval, describe the arc *a b* intersecting the line B E at F, and join O F, producing the right-angled triangle O B F. From B F cut off a part B G = $\frac{3}{4}$ (O B), and join O G. Produce A D to meet the circumference of the circle at the point M. From the right angle B in the triangle O B F, draw the straight line B H, perpendicular to its opposite side O F, and join B M.

Now, because B H is a straight line drawn from the right angle B in the triangle O B F, perpendicular to its opposite side O F; according to *Euclid, Prop.* 8, *Book* 6, B H F and B H O are similar triangles, and similar to the whole triangle O B F. But, the triangle B H O = the triangle B H M; the triangle O B M = the triangle O A B; and the diagonal A M divides the parallelogram O A B M into four similar and equal right-angled triangles. Hence, the triangles O B F, B H F, B H O, B H M, M D O, M D B, A D O, and A D B, are similar right-angled triangles.

Now, the ratios of side to side in all these triangles are *determinate*, but all these ratios are not commensurable. To explain my meaning, take the triangle O B F. The ratio of the side O F to the side O B is as 2 to 1; and the ratio of the side O F to the side B F is as 2 to the $\sqrt{3}$; that is to say, both ratios are *determinate*, but the former only is commensurable, the latter being incommensurable. Now O F, the hypothenuse of the right-angled triange O B F, is bisected at M; and O G, the hypothenuse of the right-angled triangle O B G, is bisected at N; and it is self-evident that all straight

lines drawn from the angle B O F to a point in the line B F, on either side of the line O G, will be bisected by D M, the perpendicular of, and common to, the two similar and equal right-angled triangles M D O and M D B. But, out of all the triangles that may be constructed by drawing straight lines from the angle B O F to a point in its opposite side B F, there is but one, of which the ratios of side to side are all commensurable, and that is the triangle O B G.

In the enclosed Diagram B *(see Diagram XI.)*, let O A B be an equilateral triangle, and O B the radius of the circle X, with O B bisected by the line A D, as in Diagram A. Then, bisect O D at E, and from E draw a straight line parallel to A D, to meet and terminate in the circumference of the circle X, at the point F. Produce B O to meet and terminate in the circumference of the circle X, at the point G, and join G F. From the angle B draw a tangent to the circle X to meet G F produced at H. Join B F, and from the angle F let fall the perpendicular F K. From O, the centre of the circle X, draw a straight line parallel to G H to meet and terminate in the line B H at L. With O as centre and G B as interval, describe the circle Y.

Now, because F E is a straight line drawn from the right angle F in the triangle G F B, perpendicular to its opposite side G B, according to *Euclid*, *Prop.* 8, *Book* 6, F E G, and F E B are similar triangles, and similar to the whole triangle G F B. Mr. R—— will not dispute that G F B is a right-angled triangle, since he knows as well as I do, that theoretical Geometry proves that if straight lines be drawn from the extremities of a diameter to any point in the circumference of a circle, the produced figure is a

right-angled triangle. *(Euclid, Prop.* 31, *Book* 3.*)* Again: Because B F is a straight line drawn from the right angle B in the triangle G B H, perpendicular to its opposite side G H; B F G and B F H are similar triangles, and similar to the whole triangle G B H. Again: Because F K is a straight line drawn from the right angle F in the triangle B F H, perpendicular to its opposite side B H; F K B and F K H are similar triangles, and similar to the whole triangle B F H. Hence: by theoretical Geometry the triangles G E F, F E B, G F B, B F H, F K B, F K H, G B H, and O B L are similar right-angled triangles; and it is self-evident that F B, a diagonal of the rectangle E B K F, divides the rectangle into two similar and equal right-angled triangles, F E B and F K B, which are two of these triangles.

Now, if Mr. R—— will only be good enough to find, and tell me, the ratios of side to side in all these triangles, and also tell me how many times he makes the area of the circle X to be contained in the area of the circle Y (and he cannot think me unreasonable in making this request), I will show him what follows, *and follows of necessity;* and in this way I think we may yet succeed in arriving at "*a happy state of concord*" on the ratio of diameter to circumference in a circle.

Hoping you are fast recovering from your recent attack, and with kind regards to all, in which Mrs. Smith joins me,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

Mr. R——'s Paper, *4th May*, 1868.

Mr. Smith presumes to treat me as a "precocious urchin," and sets me a trial problem! Given area of a circle P; to find the values of the sides O V R, and what $\frac{XZ}{P}$ is equal to? And he goes on to say that if π were indeterminate, it would follow of necessity that the area of P could not be contained an exact number of times in X Z.

Now, let (r) = radius of P; and (s) = radius of X Z: then, $\frac{XZ}{P} = \frac{\pi s^2}{\pi r^2} = \left(\frac{s}{r}\right)^2$, showing that the square of the quotient of s by r is the number of times that P is contained in X Z. This question, therefore, depends solely on the relation between s and r, and if π be 1 or 1000, the result will be the same. I cannot understand how Mr. Smith makes these statements. If you multiply and divide a number by the same quantity, the second operation gives you back the quantity. It is $a \times b \div b = a \times 1 = a$. Does Mr. Smith not know this so very simple and obvious fact? Surely he will see now that this proof that π is determinate is not worth a straw.

There is a paragraph in which I am accused of *wilful* misrepresentation. I may sometimes fail to see the point of Mr. Smith's statements; but he should not accuse me of *wilful* misrepresentation. In *my* ethics, even in mathematical controversy, this would be *immoral*, and wrong. I bear it; and from this (more convincingly than from his statements, it appears that $\pi = 3\frac{1}{8}$) he will see at once that I have not *wilfully* offended. I take the present statement, a = given area of circle; b = area of square, (I must have understood that b = side of square,) $\therefore \frac{a}{\frac{1}{4}\pi} = b$. If π is indeterminate, (b) must be so, of course; but B cannot be indeterminate, says Mr. Smith; therefore π cannot be so. This is a *sound* conclusion, if the premisses be all right. But how much does Mr. Smith here

assume! He postulates A. If you postulate A, then B of course depends on π. Now, I hold that you cannot *a priori* assume that B is determinate. This is simply *petitio principii.* b is the square of the diameter, and if it be the *area* of the circle that is given, the diameter depends on the ratio, and you must prove and know the ratio by other means, before you can find the diameter. Of course, in this case, b will be indeterminate, *i.e.*, you will not from $a = 5$, or 13, or 17, &c., find the diameter a terminate fraction. The value of a square depends on the value of the side of it, and seeing that the side in this case = diameter, the diameter being dependant on the ratio π, it follows that the square may not be calculable in a terminate decimal. That is simply a question of fact. I am only surprised that Mr. Smith should think this a clencher. If this convinces him, I do not wonder at his faith in $\pi = 3\frac{1}{8}$; but he may rest assured, at once and conclusively, that it is not "crammed erudition," but plain *common sense,* that prevents mathematical men from receiving such *nonsense.* I use this word in the greatest good humour, and with respect for a worthy man, who allows himself to be satisfied with, and thus exposes himself to the rude assaults of such hard-headed men as Professor de Morgan.

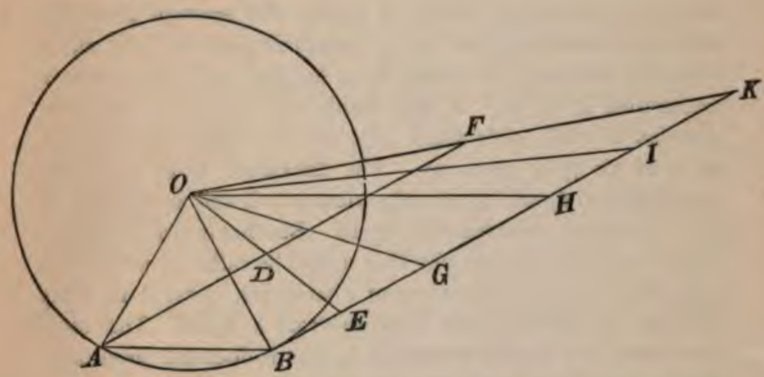
I am at a loss to understand such statements as "Geometry combined with Mathematics." Is not Geometry a *branch* of Mathematics? So I have always understood, from a boy. Since writing the last sentences, I have turned up the definition of Mathematics in the *Imperial Dictionary*, which Mr. Smith should read, and he will see my opinion verified, that *practical* Geometry can prove nothing. Practical Mathematics, including the Geometrical *branch*, is the application of previously discovered properties and relations of numbers and magnitudes. *Practical* Geometry can prove nothing, and least of all Mr. Smith's geometry. The very constructions are suggested by his supposed ratio. $3\frac{1}{8}$ has been at the construction in every case. All the constructions are really allied, and run into one another by the connection $3\frac{1}{8}$. However Mr. Smith hit on, or

found $3\frac{1}{8}$,—by a happy guess, as I once ventured to hint—or by what *seemed* a legitimate deduction, one cannot say, until the *whole* process is revealed.

All through Mr. Smith's Papers, there is a vast might of assertion. He should have known long ago that simple *assertion*, in a Mathematical question, is literally nothing. Assertion is good, in questions of testimony; but when the thing lies under my eye, as much as under his, I can judge as well as he. There is no room for assertion if we are thus on equal terms. Two Watchmakers are looking at the same watch; both are supposed to understand its relations. What if one should begin to assert—to threep—over the other, that such and such is the case; and call his neighbour names because he sees plainly that his would-be discoverer and instructor is a little dazed, and quite mistaken! I consider myself quite capable of understanding all that can be advanced by Mr. Smith, except his avowal of belief in his $\pi = 3\frac{1}{8}$, on SUCH grounds as he has hitherto condescended to reveal to the exoteric world.

Mr. Smith admits that a number may be determinate although incommensurable. I lay down this as a fact, viz., that you can often draw the line or describe the area represented by a number which cannot be expressed in *terminate decimals*. My geometry shows me to draw a line, or describe an area, represented by such numbers as $\sqrt{5}$; $\sqrt{10}$; $\sqrt{17}$; $\sqrt{3}$; $\sqrt{6}$; $\sqrt{\frac{2}{2+\sqrt{2}}}$ and many more surd or irrational numbers. I call attention to this fact, that Mr. Smith may see how irrational and arbitrary is the condition he has imposed on π, *a priori* that it must be *determinate* and expressible in terminate decimals, as well as lie between $3\frac{1}{8}$ and $3\frac{1}{5}$. A line or area may be represented by any mixed quantity, that is, by any improper fraction, as also by any proper fraction, whatever be the denominator, even though incommensurable on the decimal scale; and a surd quantity may also represent a line or area. All this is plain, and well known. Knowing these facts, I cannot see a shadow of reason for the conditions laid down for π by Mr. Smith. It is true that $3\frac{1}{8}$ obeys these conditions,

and it alone. It is the only number less than $3\frac{1}{8}$, and more than 3, that can be expressed in decimals (except $3\frac{1}{10}$, $3\frac{1}{20}$, $3\frac{1}{25}$, &c., &c., &c., which I presume must go out of court, as having no *locus standi*). But these conditions do not lie in the *nature* of *the* case, but in the necessities of Mr. Smith's case; and no more prove his π's claim to the crown, than the claim of a usurper of a throne could be proved by his laying down that the true occupant must have hair of a certain colour, and be of a certain stature, because he and no other person obeys these arbitrary conditions.



As I am jotting down desultorily a few notes, I here give one on Mr. Smith's Letter from Windermere, which sets me a problem. The construction Mr. Smith knows. Only I produce B E, and taking G, H, I, &c., &c., join these points with O, and produce A D. Now Mr. Smith says that there can be but one ratio between the sides O B and O E, and asks me to find it; and also to find the ratio between O B and B E, and between B E and O E. Of course there can be but one ratio between these sides; but Mr. Smith cannot find that ratio from his data. The datum is a right-angled triangle, one side, and a line parallel to one side cutting the hypothenuse and bisecting it. Now A D produced will bisect all the hypothenuses of all the possible triangles which have O B as their side, and you can have any number of them. Some

condition or datum must have been forgotten. Mr. Smith really asks me to find the two sides of a triangle from only one. He must see that the fact that a line parallel to the base bisects all, divides the sides proportionally in all triangles, makes his problem impossible.

But it may be that Mr. Smith uses his terms in some unusual or unique sense, a thing not uncommon with him: and thus really speaks in an unknown tongue, and gives an uncertain sound. But why these problems for the "precocious urchin?" Why not grapple with the first principles, and meet objections to them so fatal? I repeat that *practical* geometry can prove nothing; *i.e.*, if by practical geometry we mean what any dictionary describes it to be, and as I have always understood it to be. If Mr. Smith will leave out the word *practical* I could understand him, and he really does attempt to prove his $3\frac{1}{8}$ by geometry. The Buccleuch letter contained a geometrical figure ingeniously constructed, and an attempt to prove a certain point from the figure. But that is just Geometry. There is no more reason why that should be called *practical* than any other geometrical statement whatsoever, because in every problem or theorem you must *construct* a *figure*. The ONLY fault with Mr. Smith's geometry is, that it does not *prove* anything; or, rather, does not *prove* $\pi = 3\frac{1}{8}$.

It proves, too, how futile is Mr. Smith's objection to the usual method of approximating to π, inasmuch as while objecting to the comparison of an infinitesimally small line to a similarly small curve, he at once equates two squares, not infinitesimally small to a whole circle, not to an infinitesimal fragment. How does Mr. Smith justify this inconsistency?

But I have a great dislike to beating about the bush. I like to walk right up to it, and see whether it is a *hare* or a *bogle*. Therefore my rule is to put aside extraneous discussion, and stick to first principles.

I have not time to go over the complaints made by my correspondent that I have wilfully misrepresented him. I may have misunderstood, but never *wilfully* misrepresented.

The following is my solution of the very simple problem which Mr. Smith sets to the "precocious urchin":—

$O K = 2 : O B = 4 : O C = 5 : B E = 3$

$O B : O C :: O C : O F;\ 4 : 5 :: 5 : \frac{25}{4} = OF = 6{\cdot}25$

$O C : O F :: O F : O R;\ 5 : \frac{25}{4} :: \frac{25}{4} : \frac{125}{16} = OR = 7{\cdot}8125$

$OB : BC :: OR : RV;\ 4 : 3 :: \frac{125}{16} : \frac{375}{64} = RV = 5{\cdot}859375$

$OB : OC :: OR : OV;\ 4 : 5 :: \frac{125}{16} : \frac{625}{64} = OV = 9{\cdot}765625$

And, $\frac{XZ}{P} = \frac{\pi OR^2}{\pi OK^2} = \frac{OR^2}{OK^2} = \frac{125^2}{4 \times 16^2} = \frac{15625}{1024} = 15{\cdot}2587890625.$

The values of O R, R V, and O V, given here correspond to O K = 2, not to π (O K²) = 60. The values Mr. Smith finds by $\pi = 3\frac{1}{8}$ may be very correct on his theory. I do not think it worth while to test an arithmetical calculation. Not admitting $\pi = 3\frac{1}{8}$, I of course see no sense in these calculations. The values O R, R V, and O V, if correctly calculated, correspond to O K = 2, or O B = 4. If any other value of O R is to be substituted, of course the values must vary. If your datum be area, of course $\sqrt{\frac{\text{area}}{\pi}}$ = radius. But there Mr. Smith and others differ; so that his problem is no value towards a solution, though of great use as a means of beating about the bush.

Mr. Smith has given a mode of "detecting the true value of π from all counterfeits, by connecting it with some other figures in the diagram, and producing results which every other but the true value of π is utterly incompetent." And then, having constructed a number of right-angled triangles whose sides are as 3 : 4 : 5, he calls attention to the fact that the sum of the squares of the three sides is in every case = $3\frac{1}{8}$ times the square of the intermediate side. $3^2 + 4^2 + 5^2 = 3\frac{1}{8}(4^2)$. That when the sides of a right-angled triangle are in these proportions, the square of the 3 sides = $3\frac{1}{8}$ times the square of the intermediate one is of course true. But I see no connection thereby

established between $3\frac{1}{8}$ and π. There is no *nexus*. I am sure that nothing as to π is proved by this. If you cannot find, and prove the truth of the ratio otherwise, this helps nothing. I do not admit the validity or applicability of the so-called test.

BARKELEY HOUSE, SEAFORTH,

11th May, 1868.

MY DEAR SIR,

I have had many extraordinary Papers from Mr. R——, but his last—which lies before me—is the most extraordinary of all. I may probably give you a running commentary upon this remarkable Paper, but I shall be in no hurry about it, and in the meantime I shall deal with Mr. R—— in another way.

In this Paper Mr. R—— treats us with a geometrical figure. (*See Diagram*, page 193). With reference to this figure, he observes:—"As I am jotting down desultorily a few notes, I have given one on Mr. Smith's Letter from Windermere, which sets me a problem. The construction Mr. Smith knows. Only I produce B E, and taking G, H, I, &c., &c., join these points with O, and

produce A D." From this data Mr. R—— arrives at the following conclusion. "Now A D produced will bisect all the hypothenuses of all the possible triangles which have O B as their sides, and you can have any number of them." Why, my Dear Sir, you have merely to take your compasses, and small as is the figure, you may convince yourself that the line O K is not bisected at F. It appears to me that, it is only when O K the hypothenuse of the right-angled triangle O B K is parallel to A B, a side of the equilateral triangle O A B, and the generating figure of the diagram, that the lines O E, O G, O H, &c., are bisected by the line A F. It is no doubt true, that when O K is drawn parallel to A B, we may get any number of right-angled triangles, by drawing straight lines from the angle B O K to a point in its opposite side B K; but out of all these triangles—and there may be millions of them, and have all their sides expressed arithmetically—there is but one, of which the ratios of side to side, are not only arithmetically determinate, but arithmetically commensurable.*

Well, then, I am sure, my Dear Sir, that you, as referee, will not think it unfair or unreasonable on my part, to ask for some better evidence than Mr. R—— has yet afforded me—and I will give him the opportunity of furnishing it—that he is a Geometer, and *"a reasoning geometrical investigator."*

To construct the enclosed diagram A *(See Diagram XII.)*, I had nothing but my compasses, and a sheet of paper for a rule. Now, A B the diameter of the circle, is common

* In this paragraph there is a glaring blunder, into which the writer was led from the mal-construction of Mr. R——'s figure.

to the two triangles A C B and A D B; and A C B and A D B are right-angled triangles. (*Euclid: Prop.* 31; *Theorem, Book* 3). Hence: A C^2 + C B^2 = A D^2 + D B^2, and, it is self-evident, that this equation = area of a circumscribing square to the circle. Again: D E is a straight line drawn from the right angle D in the triangle A D B, perpendicular to its opposite side A B, therefore, according to *Euclid: Prop* 8, *Book* 6, D E A and D E B are similar triangles, and similar to the whole triangle A D B.

Now, the triangles A C B and A D B are commensurable right-angled triangles, that is to say, the ratio of side to side in both, are not only commensurable, but determinate and finite: and since the triangles A D B, D E A, and D E B are similar right-angled triangles, the ratios of side to side in these triangles, must be the same. Hence: The straight lines A C, C B, A D, D B, A B, A E, B E, and D E, can be expressed arithmetically in the terms of their ratios to each other, or in other words, the arithmetical values of all these lines can be expressed in whole numbers. But, we may construct other triangles within a circle, having the diameter of the circle as a common hypothenuse, so that the ratios of side to side shall not only be commensurable, but determinate and finite. Such are the right-angled triangles F H G and F K G in diagram B *(see Diagram XIII.)*, and since K L is a straight line drawn from the angle K in the triangle F K G, perpendicular to its opposite side F G, K L F and K L G are similar triangles, and similar to the whole triangle F K G. Hence: The straight lines F H, H G, F K, K G, F G, F L, G L, and K L can be expressed arithmetically in the terms of their ratios to each other; or in other words, the arithmetical values of all these lines

can be expressed in whole numbers. I could give other examples, but surely this must be unnecessary.

Now, Mr. R—— may tell us that straight lines may be drawn from A and B to other points in the circumference of the circle than C and D in diagram A, and that the equation $A C^2 + C B^2 = A D^2 + D B^2 =$ area of a circumscribing square to the circle: and, he may also tell us that straight lines may be drawn to other points than H and K in diagram B, and that the equation $F H^2 + H G^2 = F K^2 + K G^2 =$ area of a circumscribing square to the circle; and then *boldly assert that this proves nothing.* I admit the premisses, but deny the conclusion. Mr. R—— will *know better* than tell us we cannot give values to all the straight lines in the diagram, with arithmetical exactness! Were he to do so, it would be tantamount to telling us that he is gifted with the capacity to prove a *negative.* Well, then, I shall ask two favours of Mr. R——, and I am sure, my Dear Sir, you will be of opinion, that there is nothing unfair or unreasonable in my asking these favours!!

That Mr. R—— may have no excuse for saying he misunderstands me, I quote the following from a living writer on mathematics, and a compiler of Euclid (J. R. Young). The quotation occurs in his commentary on Euclid's first book. "*A proposition is called a problem, when the thing proposed is an operation to be performed—a construction to be effected: its object is a practical result, to be brought about by a suitable disposal and combination of the elementary materials furnished by the postulates. A proposition is called a theorem, when the thing proposed is a truth to be demonstrated; and for this demonstration the elementary materials are furnished*

by the axioms. It usually happens, however, that the proof of a theorem requires the previous introduction of certain lines and constructions; and hence it is that Euclid commences his first book with problems instead of theorems."

Now, I have no doubt Mr. R—— will agree with J. R. Young's definitions of a problem and a theorem. Well, then, the construction of the geometrical figures represented by the enclosed diagrams are *problems*. Let Mr. R—— solve these problems, and shew the method of construction. To demonstrate that all the straight lines in the diagrams may be expressed with arithmetical exactness, and in whole numbers, are *theorems*. Let Mr. R—— solve these theorems. I may tell Mr. R——, that if he read my early Letters with care, he would have no difficulty in dealing with diagram A, but I do not think I have in any part of our correspondence, given him a clue to the construction of diagram B.

Mr. R—— seems to have felt rather keenly the charge I brought against him of wilful misrepresentation. Now, my Dear Sir, what could I suppose? The facts are indisputable! Mr. R—— DID substitute the words *side of square* for the words *area of square*, and then applied my reasoning to the altered premisses. How could I suppose that this was not done wilfully? It appears to me that the very nature of our controversy forbad such a supposition. I am willing, however, to accept Mr. R——'s explanation when he says:—"*I must have understood that b = side of square.*" But what follows? Does not this prove, that Mr. R—— reads my Letters with a degree of *wilful* carelessness, and so fails to make himself master of my statements, arguments, and conclu-

sions? Can this be fair or reasonable on the part of one, who "*presumes to treat me*" as "*a little dazed and quite mistaken!*"

I have as great a dislike as Mr. R—— to "beating about the bush," and I think he will not charge me with introducing "*extraneous discussion*" into this communication. I have started from "*first principles*," and proved these principles by Euclid. Well, then, let Mr. R—— face the enclosed diagrams; he will find they are not "*bogles*"; and if he grapple with them, and solve my problems and theorems, I shall be prepared to shew him what follows, and the important part played by commensurable right-angled triangles, in the discussion of the questions upon which we have been so long at issue. Mr. R—— must not imagine he can get out of the difficulty, by declining to comply with my request, and contenting himself with telling me: "*My rule is to put aside extraneous discussion and stick to first principles.*" Will Mr. R—— be good enough to explain, in what sense he understands the term, "*first principles?*"

I was glad to hear from your Letter to Mrs. Smith, that you are recovering from your recent attack, and with kind regards to you and your family circle, in which Mrs. Smith joins me,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

MR. R——'s PAPERS, *May 9th and 15th*, 1868.

One Communication.

May 9th.

It is a very easy matter to do all that Mr. Smith propounds. Having GB = 8, GE = 5, EB = 3. We set $EF = \sqrt{GE \cdot EB} = \sqrt{15} : BF = \sqrt{GB \cdot BE} = \sqrt{24} = 2\sqrt{6} : GF = 2\sqrt{10}$: $BH = 8\sqrt{\frac{3}{5}} : GH = 16\sqrt{\frac{2}{5}} : KH = 3\sqrt{\frac{3}{5}} : FH = 6\sqrt{\frac{2}{5}}$: $OL = \frac{1}{2} GH = 8\sqrt{\frac{2}{5}} : BL = \frac{1}{2} BH = 4\sqrt{\frac{3}{5}}$. Ratios as follows:—

$$\frac{GF}{GE} = \frac{2\sqrt{10}}{5} = 2\sqrt{\frac{2}{5}} = \frac{GH}{GB} = \frac{OL}{OB} = \frac{FB}{FE} = \frac{BH}{BF} = \frac{BG}{GF} = \frac{FH}{FK} = \frac{BT}{BK}$$

$$\frac{GF}{FE} = \frac{2\sqrt{10}}{\sqrt{15}} = 2\sqrt{\frac{2}{3}} = \frac{GH}{BH} = \frac{OL}{BL} = \frac{FB}{BE} = \frac{BH}{FH} = \frac{BG}{BF} = \frac{FH}{KH} = \frac{BF}{FK}$$

$$\frac{GE}{EF} = \frac{5}{\sqrt{15}} = \sqrt{\frac{5}{3}} = \frac{GB}{BH}, \text{ \&c., \&c.}$$

And $\frac{Y}{X} = 4$.

What I am surprised at is that Mr. Smith makes so much use of straight lines to establish his π, when he won't allow Mathematicians to do the same.

May 15*th.*

I have run through Mr. Smith's communication of the 11th. He asks what I mean by first principles. I may answer, any truth that lies at the *foundation* of a process of reasoning. Something that must be true, in order that a certain argument may be valid. For instance: it is one of Mr. Smith's first principles that $\frac{\text{circum.}}{\text{diam.}}$ = a number that is expressible, not merely as a mixed number, but by decimals that terminate. It is another that this number is *less* than 3·2. If both these propositions be true, then I am not aware at present that we could resist the conclusion that the number is 3·125. Now is it strange that I keep my finger on these two propositions and say to Mr. Smith *prove them true?* I have, from mere courtesy, solved a few simple extraneous problems which Mr. Smith has set me, and pointed out, as I do in the notes accompanying this, some oversights and mistakes. But I am not going to do so any more. I have not time to waste in guesses at Mr. Smith's meaning, or in trying to show him, so convinced and complacent, that he traverses another of his first principles, by comparing surfaces bounded by straight lines with the circle, while he will not allow a comparison of the chord with the curve it subtends. It will save his time and mine, if, when he has any strong reason, he will bring it forward straight, and if something lies behind, or hangs upon, or follows from, something he will solve his own problem and apply them at once. The statements of his last letter are unintelligible, and I see nothing but absurdity in them. Perhaps he could make them plainer to a "precocious urchin:" but it is not worth his while, as it will all resolve itself in 7 : 1, and $7^2 + 1^2$ = area of circle. Nothing that Mr. Smith can say or show by any variety of expressions or construction can add anything to the Buccleuch

demonstration. *That* is as strong as any succeeding arrangement, and no succeeding arrangement adds anything of demonstration to what we find there. I may assure Mr. Smith that no man who *knows* anything of Geometry, will fail to wonder at his asking assent to such a conclusion as $3\frac{1}{8} = \pi$, on *such grounds*, He can only say there is *no proof*. Mr. Smith may therefore give me up. It is simply wasting his time to go on in this track, with one whom he asks to prove to *him* whether he understands Geometry or not. The best advice I can think of is that Mr. Smith should master the usual methods of approximation to π, and like Mr. Harbord and others great in arithmetic, test the results. In such lengthy calculations mistakes are imminent, and the *more* able and experienced calculators go through and test the results by the various processes, the more certain we become of their accuracy.

Mr. JAMES SMITH *to* Mr. J—— S——.

BARKELEY HOUSE, SEAFORTH,
16th May, 1868.

MY DEAR SIR,

I am without any of Mr. R——'s Papers since that in reply to my Letter of the 27th April, and my first Letter from Windermere. I should have had a reply to my second Letter from Windermere ere this, and by this

morning's post, might have had one to my last Letter. Is it Mr. R——'s intention to drop the controversy? Be this as it may, being a little surprised at Mr. R——'s silence, I was led to read over my two or three last Letters; and what was my astonishment, on reading over that of the 11th instant, to find I had fallen into a most glaring blunder! A blunder so glaring, that I should be surprised if Mr. R—— could have thought otherwise, than that I must have been "*a little dazed*" at the time I penned it.

Now, my dear Sir, "*in my ethics, it would be immoral and wrong*" if, having committed a fault, and done any one an injustice—however unintentionally—I hesitated for a moment to admit it, or did not endeavour to make atonement to the best of my ability.

Well, then, in the second paragraph of my last Letter, I have done Mr. R—— an unintentional injustice, and I must ask *you*, as an especial favour, to kindly forward this Letter of explanation to him, with my humblest apology, and a request that he will be pleased to substitute the following for the paragraph in question :—

In this Paper, Mr. R—— gives us a geometrical figure, based on that enclosed in my first Letter from Windermere, and with reference to this figure he observes:— "*As I am jotting down desultorily a few notes, I have given one on Mr. Smith's Letter from Windermere, which sets me a problem. The construction Mr. Smith knows. Only I produce* BE *and taking* G, H, I, *&c., &c., join these points with* O *and produce* AD. *Now, Mr. Smith says that there can be but one ratio between the sides* OB *and* OE, *and asks me to find it, and also to find the ratio between* OB *and* BE, *and between* BE *and* OE. *Of*

course there can be but one ratio between these sides; but Mr. Smith cannot find that ratio from his data. The datum is a right-angled triangle, one side, and a line parallel to one side cutting the hypothenuse and bisecting it. Now, A D *produced will bisect all the hypothenuses of all the possible triangles which have* O B *as their side; and you can have any number of them. Some condition or datum must have been forgotten. Mr. Smith really asks me, to find the two sides of a triangle from only one. He must see that the fact that a line parallel to the base bisects all, divides the sides proportionaly in all triangles, makes his problem impossible.*" Now, it is no doubt true, that since A F is parallel to B K, it follows, that we may get any number of right-angled triangles, by drawing straight lines from the angle B O K to a point in its opposite side B K, and that the hypothenuses of all these triangles will be bisected by the line A D produced; but, out of all these triangles—and there may be millions of them, and have all their sides expressed arithmetically determinate—there is but one, of which the ratios of side to side, are not only arithmetically determinate, but arithmetically commensurable. It is no doubt true, that in my first Letter from Windermere, "*a certain datum or condition was forgotten;*" and had I not forgotten, and unintentionally omitted to give this datum or condition, Mr. R—— might have asserted that the solution of my problem is impossible, and that "*practical geometry can prove nothing;*" but he certainly could not have proved these assertions.

Well, then, as a fair and candid opponent, Mr. R—— will surely permit me to rectify a blunder, by kindly

substituting the foregoing for the second paragraph in my last Letter.

Now, "*in my ethics, even in mathematical controversy, it would be immoral and wrong*" to wilfully deceive an opponent, or cunningly set a trap to catch him to serve a purpose; and I cannot subscribe to the doctrine that "*the end justifies the meaus*,"—if the means be immoral and wrong—although the end aimed at may be praiseworthy; hence, as soon as it occurred to me that Mr. R—— might be deceived by my first Letter from Windermere, I wrote that of the 4th instant, in which I have certainly made my meaning unmistakeably intelligible; and I am sure that, with the diagrams enclosed in that Letter before him, it will be as inexplicable to Mr. R—— as it is to myself, how I could have fallen into so gross and stupid a blunder as that in my last Letter, which has obliged me to enter into this explanation, and make an apology to Mr. R——.

The diagram in the margin *(see Diagram, page 185)* is a copy of that enclosed in my first Letter from Windermere. The following is the datum or condition, unintentionally omitted in that Letter, and which should have followed the description of the method of constructing the diagram:—

Now, if from the angle O we draw a straight line parallel to A B to meet B E produced at K, O B K will be a right-angled triangle, of which the side O K is to the side O B as 2 to 1; that is, of which the hypothenuse is to the perpendicular in the ratio of 2 to 1; and it is self-evident, that if we draw a number of straight lines from the angle B O K to points in its opposite side B K, these lines will produce a series of right-angled triangles,

of which O B is a side; but of all these triangles, there is but one, of which all the sides are not only determinate, but commensurable, and that triangle is represented by the triangle O B E.

Well, then, the simple object of my first Letter from Windermere, was to suggest to Mr. R—— the relations that exist between a right-angled triangle of which the sides are 3, 4, and 5, and an equilateral triangle. Mr. R—— will remember that in more than one of my early Letters, I directed his attention to the right-angled triangle of which the sides are 3, 4, and 5, calling it the *primary* commensurable right-angled triangle, since it is the first commensurable right-angled triangle of which the arithmetical values of the sides can be expressed in whole numbers; and the triangle of which the sides that contain the right angle are in the ratio of 3 to 4, has played an important part in almost every Letter of mine that has fallen into the hands of that gentleman. Now, my second Letter from Windermere affords the clearest evidence that I had no wish or intention to deceive Mr. R——, and even proves that I was not really ignorant of the geometry of his figure; what I merely intended to suggest by the former, I was—from the omission referred to—obliged to prove in the latter, by making the sides that contain the right angle in the triangle O B G, in diagram No. 1, in the ratio of 3 to 4, by construction. This obliged me to give the diagram No. 2, in which no arithmetical symbol enters into the construction. The construction of the figure is accomplished by Geometry. Mr. R—— may call it theoretical, speculative, or practical geometry, by which the figure is produced. I care not which. In his last Paper, he says:—

"*I repeat that practical Geometry can prove nothing.*" But I can assure *you*, and, through *you*, Mr. R——, that much may be proved by this geometrical figure, in spite of his assertion. The *Imperial Dictionary*—which Mr. R—— recognises as an authority—does not adopt and establish his view that "*practical Geometry can prove nothing.*"

Had Mr. R—— replied to my last Letter from Windermere with his usual punctuality, mine of the 11th inst. would never have fallen into Mr. R——'s hands. I should have been spared the mortification of making a stupid mistake—hardly conceivable to have been made by any one but a *daft*—and the necessity of making this explanation; and all of us would have been spared some trouble that might, and should have been, avoided. To you, my dear Sir, I owe an apology, for giving you unnecessary trouble, and frankly make it, but at the same time I feel assured, that you will as frankly pardon me.

Mrs. Smith joins me in the hope that you have quite recovered from your recent attack, and with our united kind regards to yourself, Mrs. and Miss S——,

Believe me, my dear Sir,

Most truly yours,

JAMES SMITH.

J—— S——, Esq.

MR. JAMES SMITH *to* MR. J—— S——.

BARKELEY HOUSE, SEAFORTH,
19th May, 1868.

MY DEAR SIR,

I am in receipt of your esteemed favour of yesterday, with its enclosure, and am glad to find you are improving, and hope to hear soon that you have quite recovered from your recent attack of illness.

I cannot help thinking that you will be wondering how frequently, in the course of our correspondence, Mr. R——'s Papers and my Letters, have crossed each other. Be this as it may, however, I can assure you that Mr. R——'s last Paper has had my most careful attention. That part of it which was written on the 15th inst., commences as follows:—"*I have run through Mr. Smith's communication of the* 11*th. He asks me what I mean by first principles. I may answer:—Any truth that lies at the foundation of a process of reasoning. Something that must be true, in order that a certain argument may be valid. For instance:—It is one of Mr. Smith's first principles that* $\frac{\textit{circumference}}{\textit{diameter}}$ *= a number that is expressible, not merely as a mixed number, but by decimals that terminate. It is another, that this number is less than* 3·2. *If both these propositions be true, then I am not aware, at present, that we could resist the conclusion that this number is* 3·125. *Now, is it strange that I keep my finger on these two propositions, and say to Mr. Smith, prove them true?*

Now, my dear Sir, I deny Mr. R——'s assertions, that it is one of my first principles that $\frac{\text{circumference}}{\text{diameter}}$

= a number that is expressible, not merely as a mixed number, but by decimals that terminate; and, that it is another of my first principles, that this number is less than 3·2. On the contrary, I maintain, that we must resort to first principles, and by means of constructive or practical geometry, prove both. In geometry the first principles are the definitions, postulates, and axioms, and all the problems and theories in Euclid are deduced from these by practical geometry. It is one of Mr. R——'s first principles that we can approximate to the value of π by means of multilateral inscribed polygons to a circle of radius 1. I deny that this is a first principle, and ask Mr. R—— for his proof. He may tell me that he has never said this is one of his first principles. What then does he mean when he says: "*The best advice I can think of is, that Mr. Smith should master the usual methods of approximation to* π, *and like Mr. Harbord and others great in arithmetic, test the results. In such lengthy calculations mistakes are imminent, and the more able and experienced calculators go through and test the results by the various processes, the more certain we become of their accuracy.*"

Well, then, you know, my dear Sir, that in the affairs of common life, we may frequently be in a position to prove what a thing cannot be, and yet be unable to prove what it is. And so, in Mathematics—and I admit that geometry is a branch of mathematics—we may prove by a certain process what a thing cannot be, and yet be unable to prove, by the same process, what it is. For instance:—We may prove, by a very simple process, that π cannot be greater than 4, nor less than 3, since the former supposition would make the circumference of

a circle of diameter 1 greater than the perimeter of its circumscribing square, and the latter less than the perimeter of its inscribed regular hexagon; but this process neither proves what π is, nor whether it be a determinate or an indeterminate quantity.

Now, I maintain that there is a process by which we can prove what π cannot be, and yet be unable, by this process, to prove what π is. In other words, I maintain that there is a process by which we can prove that π cannot be indeterminate, and yet, be unable to prove by this process, what π is. I must give you the proof.

The enclosed diagram, No. 1 *(See Diagram XIV.)*, is a fac-simile of the diagram No. 2, contained in my second Letter from Windermere, with the following additions. From the angle O, in the equilateral triangle O A B, the generating figure of the diagram, let fall a perpendicular, bisecting its opposite side A B at M. From the angle M draw a straight line parallel to A D, to meet and bisect the line D B at N. With O as centre, and O N as interval, describe the circle Y; and with O as centre, and O M as interval, describe the circle X.

Now, E B = B L, by construction. But, O N = E B; therefore, O N = B L, and O N : O B : : 3 : 4. Hence: If O B, the radius of the circle Z, = 4; O N, the radius of the circle Y, = 3; and, O M, the radius of the circle X = $\sqrt{12}$; or, if O B = $\sqrt{12}$, O N = $\sqrt{6 \cdot 75}$, and O M = 3; therefore, the triangle O N M and M N B are similar triangles, and similar to the whole triangle, O M B; and it follows, that the radii of the circles Y, X, and Z, although all determinate, can never be all commensurable; therefore, O M B is an incommensurable right-angled triangle,

Well, then, let the letters which represent the circles denote the arithmetical value of their areas. Then: Y : X : : 3 : 4; and, Y : Z : : 3^2 : 4^2. We can prove these facts by means of any hypothetical value of π intermediate between 3 and 4, so that it be finite and determinate, and it follows of necessity, that whatever be the value of π *it cannot be an indeterminate arithmetical quantity*. What Mr. R—— may say to this, I know not. I give him the facts; if he dispute them, let him disprove them.

The enclosed diagram, No. 2 *(See Diagram XV.)*, is a fac-simile of the diagram, No. 2, contained in my Letter of the 4th inst., with the following additions. From the angle O, in the equilateral triangle O A B, let fall a perpendicular, bisecting its opposite side, A B at M. From the right angle M, in the triangle O B M, draw a straight line perpendicular to its opposite side O B, dividing the triangle into two similar right-angled triangles, O N M and M N B, which are similar to the whole triangle, O M B. From the angle F, in the triangle G F B, draw a straight line through O, the centre of the circle X, to meet and terminate in the circumference at the point P, and join G P and B P, producing the rectangle G P B F. Produce F E to meet P B, a side of the rectangle, at the point T.

Now, because F E is parallel to B K, F T is parallel to B H; and because G F is parallel to P B, F H is parallel to B T, and it follows, that F T = B H, and F H = B T, and the rhomboid, F T B H, is divided by the diagonal F B into two similar and equal right-angled triangles, F B T and B F H; and, because B F is a straight line drawn from the right angle B, in the triangle G B H, perpendicular to its opposite side, G H, G F B,

and B F H are similar triangles, and similar to the whole triangle, GBH ; and GFB and FBT are similar triangles.

Now, if G B = 8, G E = 5, and E B = 3, as Mr. R—— puts it. But, because B L = E B and 2(B L) = B H, it follows, that B H and F T = 6, G H = 10, G F = 6·4, F H and B T = 3·6, F B = 4·8, and B L = 3.

Hence:

G B : B H : : 8 : 6.
G F : F B : : 6·4 : 4·8.
B F : F H : : 4·8 : 3·6.
F B : B T : : 4·8 : 3·6.
G B : G H : : 8 : 10.
G F : G B : : 6·4 : 8.
B F : B H : : 4·8 : 6.
F B : F T : : 4·8 : 6.

and it follows, that G H : G B : : O L : O B, that is, 10 : 8 : : 5 : 4; and, G B : B H : : O B : B L, that is, 8 : 6 : : 4 : 3; therefore, G B H, B F G, B F H, F B T, and O B L are similar triangles, and have the sides that contain the right angle in the ratio of 3 to 4. Mr. R—— will hardly think it necessary that I should go into the figuring, to prove that in all these triangles the sum of the squares on the sides that contain the right angle, are equal to the square on the side that subtends the right angle.*

I do not think Mr. R—— can dispute the ratios, and I think he will find that they do not harmonize with the ratios as given in his last Paper, which lies before me.

I will now direct your attention, and through you, the attention of Mr. R——, to an extraordinary anomaly in mathematics.

The triangle G F B is a commensurable right-angled

triangle, having the sides that contain the right angle in the ratio of 3 to 4, and when the side G B = 8, G F = 6·4, and F B = 4·8, that is, G F = $\frac{4}{5}$ (G B), and F B = $\frac{3}{5}$ (G B). Now, by Euclid: Prop. 8, Book 6: in a right-angled triangle (G F B), if a perpendicular (F E) be drawn from the right angle to the opposite side, the triangles, F E G and F E B, on each side of it, are similar to the whole triangle, and to each other. But, in the triangle G F B, G F = $\frac{4}{5}$ (G B), and when G B = 8 = 6·4; and G B is divided into two parts, G E and E B, which are in the ratio of 5 to 3; therefore, when G B = 8, G E = 5. Now, since, by theoretical geometry, G E F and G F B are similar triangles, $\frac{4}{5}$ (G F) should equal G E. But, G F = 6·4, when G B = 8, and $\frac{4}{5}$ (G F) = $\frac{4}{5}$ (6·4) = 5·12, and is not equal to G E. How is this? Is mathematics at fault?*

Now, if, instead of charging me with such like nonsense, as that I hold it to be a first principle, "that $\frac{\text{circumference}}{\text{diameter}}$ = a number that is expressible, not merely

* There is a fallacy in these paragraphs. The reasoning is based on the assumption that O L and G H are parallel lines, which would make the triangles O B L and G B H similar right-angled triangles. But, O L and G H are not parallel lines. I did not discover the fallacy until some time after the publication of "Euclid at Fault." Strange to say, neither Mr. R—— nor any of the numerous Mathematicians who attacked that pamphlet discovered the fallacy. On sending Mr. R—— a copy, I received the following communication from him:—

16th July, 1868.

My dear Sir,

I have to thank you for the pamphlet and the accompanying letter received to-day.

In my notes I pointed out where you are wrong. K H is not $\frac{4}{5}$ (K B), by construction. Neither is it self-evident that B F = B T, the half of B M. Your construction, as I pointed out, is K F = 5, and B F = 3: and this gives you H F = $\sqrt{15}$, K H = $\sqrt{40}$, and H B = $\sqrt{24}$. These are the values

as a mixed number, but by decimals that terminate," Mr. R—— will give this Letter his careful consideration, I yet think there is hope of our arriving at a "happy state of concord," on the value of π, and the ratio of diameter to circumference in a circle.

Mrs. Smith joins me in kind regards to yourself, Mrs. and Miss S——, and hoping to find, when I next hear from you, that you have quite recovered,

Believe me, my dear Sir,

Very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. R——'s PAPER, *May 22nd*, 1868.

There is one oversight in Mr. Smith's Paper of the 19th. On page 4 he makes E B = B L, *by construction*. Then, since E B : E F : : B L : B O, E F = B O. That is, E F = O F or the hypothenuse = one of the sides! Mr. Smith forgot that in his Letter of the 4th, he took E, making O E = ¼ O B,

of the sides of the triangles, as I shewed you before. That H F is = $\sqrt{15}$ is plain from a proposition in the third Book of Euclid, and does not depend on the similarity of triangles: and if H F = $\sqrt{15}$, and the 47th proposition of Euclid's first Book be admitted, then K H = $\sqrt{40}$, and H B = $\sqrt{24}$.

B F is not equal to B T. You first fix T by drawing O T parallel to K M, for your construction just amounts to that. How then can you assume that B T = B F?

All this I shewed you before, but you have taken no notice of it.

You will find that all who look at your pamphlet will say the same as I have done.

I am, my dear Sir, Yours very truly,

R——

In a short reply I observed:—"Your assertions would resolve the following Postulate of Euclid into a gross absurdity. 'Let it be granted, that a circle may be described from any centre, with any interval from that centre.'" With this our correspondence terminated.

raised E F perpendicular, and drew O L *parallel* to G F produced to H. This fixes B L at what I gave it, $\frac{1}{2}$ B H $= \frac{4}{5}\sqrt{15} = 4\sqrt{\frac{3}{5}}$. The difference between us arises from the fact that Mr. Smith has forgotten what follows from his own constructions. If the construction be B L $= 3$, and G H be drawn parallel to O L, then B E $= 2\cdot88$, not 3, and G E $= 5\cdot12$. If, on the other hand, the construction be O L, parallel to G H, the direction of G H being fixed by E F, G E being $= 5$, then, the values are different: they are as I gave them, if I committed no mistake. If G E $= 5$, E B $= 3$ and B L 3, and O L joined, then, O L is *not parallel* to G H. For if so, B O L and B E F are similar, and B O $=$ E F $=$ O F as above.

Hence Mr. Smith's difficulty (in page 7) to prove $5 = 5\cdot12$. It is not mathematics at fault, but simply an oversight in mixing up two constructions.

Again: pp. 4 and 5, "$y : x :: 3 : 4$ and $y : z :: 9 : 16$. He can prove these facts by means of any hypothetical value of π intermediate between 3 and 4, *so that it be finite and determinate,* and it follows of necessity, that π cannot be indeterminate. What Mr. R—— may say to this I know not, I give him the facts, &c." Why does Mr. Smith introduce the words "*so that it be finite and determinate*"? You can prove that circles are as the squares of their radii by any *hypothetical value of* π, *whether* or *not* it be finite and determinate. Let $\pi =$ any hypothetical quantity, then $y : x :: 9\pi : 12\pi :: 3 : 4$. The *arithmetical value of* π is *not* necessary to the proof of this ratio. Hence, the conclusion does not follow from the premisses, and still waits for its much needed proof. I am, with Mr. Smith's proof, as the hungry man who dreams of food, and awakes, and lo! there is none.

BARKELEY HOUSE, SEAFORTH,
23rd May, 1868.

MY DEAR SIR,

I have caught Mr. R—— at last, and I venture to tell *you*, that my Letter of the 19th inst. will astonish that gentleman more than he was ever before astonished by mathematics; and that we shall not have him again reiterating the ***absurd assertion***, "that practical geometry can prove nothing." It is not necessary I should wait Mr. R——'s answer to that communication before dipping a little deeper into the mathematical anomaly there referred to; and as I am not going to Liverpool to-day, I commence this Letter, but shall not post it before Monday, by which time I may have his reply.

Well, then, before I conclude this Letter, I shall prove that, the 8th Proposition of Euclid's 6th Book, ***is not of general and universal application;*** that is to say, I shall prove, that there are right-angled triangles, in which a straight line drawn from the right angle perpendicular to its opposite side, does not make the triangles on each side of it similar to the whole triangle, and to each other, but, before doing so, I shall direct your attention, and through you the attention of Mr. R——, to some geometrical facts; and then, I shall put two questions to, and ask a favour of, Mr. R——.

In the enclosed diagram No. 1 *(see Diagram X VI.)*, let A and B be two points dotted at random. Join AB. Post 1. Produce AB to C, making AC equal to 5 times AB. Post 2. With A as centre, and any interval greater than the half of A C describe the circle X; and with C as centre and the same interval describe the circle Y. Post 3. The

circumferences of these circles intersect each other at the points *a* and *b*. Join these points. The line *a b* bisects the line C A at the point O. With O as centre, and O A or O C as interval, describe the circle Z. With C as centre, and C B as interval, describe the arc B D, and join C D and A D, producing the right-angled triangle C D A. (Euclid: Prop. 31, Book 3). With A as centre, and A D as interval, describe the arc D E, and join E D, producing the triangle E A D, which is an *isosceles* but not an *equilateral* triangle. From the angle A draw a straight line at right angles to C A, and therefore tangental to the circle Z, to meet a perpendicular let fall from the angle D in the triangle C D A, at the point G, and join F G and F D. From O, the centre of the circle Z, draw a straight line parallel to C D to meet and terminate in the line A G at the point H, and join D H. From the angle D, in the triangle C D F or E D F, draw a straight line through the point O, the centre of the circle Z, to meet and terminate in the circumference of the circle at the point K, and join A K and C K, producing the rectangle C K A D. Produce C D to meet A G produced at the point L.

Now, my dear Sir, you will observe that I might have adopted a different method of construction. For instance:—Instead of saying, with A as centre and A D as interval, describe the arc D E, I might have said, with A as centre and $\frac{3}{5}$(A C) as interval, describe the arc E D; and you will see that the result would have been the same.

Now, C D = C B = $\frac{4}{5}$(C A); and A D = A E = $\frac{3}{5}$(C A), by construction. O H is parallel to C D and therefore parallel to C L. K A is parallel to O H and C L. D G is parallel to C A, and D F is parallel to A L,

D A is parallel to C K, and at right angles to C D. C D A and A K C are similar and equal right-angled triangles, and C A, the diameter of the circle Z, is the hypothenuse of, and common to, the two triangles. All these facts arise out of the construction of the diagram. Hence: A K C, C D A, C D F, D F A, A D G, A D L, D G L, O A H, and C A L are similar triangles, and in all these triangles, the sides that contain the right angle are in the ratio of 3 to 4, by construction.

Let C A, the diameter of the circle Z, = 8.

Then:

$$
\begin{array}{ll}
CD = KA = CB = \tfrac{4}{5}(CA) & = 6{\cdot}4. \\
DA = CK = AE = \tfrac{3}{5}(CA) & = 4{\cdot}8. \\
CF = \tfrac{4}{5}(CD) & = 5{\cdot}12. \\
DF = AG = \tfrac{3}{5}(CD) & = 3{\cdot}84. \\
FA = DG = CA - CF = \tfrac{3}{5}(DA) & = 2{\cdot}88. \\
GL = \tfrac{3}{4}(DG) = AL - AG & = 2{\cdot}16. \\
CL = \tfrac{5}{4}(CA) & = 10. \\
OH = \tfrac{1}{2}(CL) & = 5. \\
OA = \tfrac{1}{2}(CA) & = 4. \\
AH = \tfrac{3}{4}(OA) = \tfrac{1}{2}(AL) & = 3. \\
HG = AG - AH = 3{\cdot}84 - 3 & = {\cdot}84. \\
FE = AE - AF = 4{\cdot}8 - 2{\cdot}88 & = 1{\cdot}92. \\
CE = CA - EA = 8 - 4{\cdot}8 & = 3{\cdot}2.
\end{array}
$$

Now, D F E is a right-angled triangle; therefore, by Euclid: Prop. 47: Book 1: $DF^2 + FE^2 = DE^2$; therefore, $3{\cdot}84^2 + 1{\cdot}92^2 = 14{\cdot}7456 + 3{\cdot}6864 = 18{\cdot}432 = DE^2$; and, since 18·432 is not a square number, $DE = \sqrt{18{\cdot}432}$, and D F E is an incommensurable right-angled triangle. But, C F D is a right-angled triangle; therefore, $CF^2 + FD^2 = CD^2$; therefore, $5{\cdot}12^2 + 3{\cdot}84^2 = 26{\cdot}2144 + 14{\cdot}7456 = 40{\cdot}96 = CD^2$; therefore, $\sqrt{40{\cdot}96} = 6{\cdot}4 = CD$, and C F D is

a commensurable right-angled triangle. Now, D E C is an oblique-angled triangle, and by Euclid: Prop. 12: Book 2: $DE^2 + EC^2 + 2(EC \cdot EF) = CD^2$. But, $DE^2 = 18{\cdot}432$: $EC = 3{\cdot}2$; therefore, $3{\cdot}2^2 = 10{\cdot}24 = EC^2$: $EF = 1{\cdot}92$; therefore, $2(EC \cdot EF) = 2(3{\cdot}2 \times 1{\cdot}92) = 2(6{\cdot}144) = 12{\cdot}288$; therefore, $(DE^2 + EC^2 + 2(EC \cdot EF)) = (18{\cdot}432 + 10{\cdot}24 + 12{\cdot}288) = 40{\cdot}96 = 6{\cdot}4^2 = CD^2$.

Again: D G H is a right-angled triangle, therefore, $DG^2 + GH^2 = DH^2$. But, $DG = 2{\cdot}88$ and $GH = {\cdot}84$, therefore, $2{\cdot}88^2 + {\cdot}84^2 = 8{\cdot}2944 + {\cdot}7056 = 9 = DH^2$; therefore, $\sqrt{9} = 3 = DH$; therefore, $DH = HA$; therefore, $DH = \frac{1}{2} AL$, and the triangle D G H is a commensurable right-angled triangle. But, D G A is a right-angled triangle; therefore, $DG^2 + GA^2 = DA^2$, and $DG = 2{\cdot}88$, and $GA = 3{\cdot}84$; therefore, $DG^2 + GA^2 = 2{\cdot}88^2 + 3{\cdot}84^2 = 8{\cdot}2944 + 14{\cdot}7456 = 23{\cdot}04 = DA^2$. Now, D H A is an oblique-angled triangle; therefore, $(DH^2 + HA^2 + 2(AH \cdot HG)) = DA^2$. But, $DH^2 = 3^2 = 9$, and $HA^2 = 3^2 = 9$, and, $2(AH \cdot HG) = 2(3 \times {\cdot}84) = (2 \times 2{\cdot}52) = 5{\cdot}04$; therefore, $(DH^2 + HA^2 + 2(AH \cdot HG)) = (9 + 9 + 5{\cdot}04) = 23{\cdot}04 = DA^2$; therefore, $\sqrt{23{\cdot}04} = 4{\cdot}8 = DA$. Hence: D G H is a right-angled triangle, of which the sides that contain the right angle, are in the ratio of 7 to 24, by construction.

Again: in the figure OAHD, OD = OA, for they are radii of the circle Z, and I have proved that DH = AH; therefore, the figure OAHD is bisected by O H; therefore, O A H and O D H are similar and equal right-angled triangles, and have O H as their common hypothenuse. Now, it is self-evident, that by producing the line *ab* to meet the line C L at the point P, C L will be bisected at P, and that by joining PA and PH we should get 3 other right-angled

triangles, not only similar to the triangles O A H and O D H, but similar to all the right-angled triangles in the geometrical figure represented by the diagram, the triangles D F E and D G H excepted. Now, $3\frac{1}{8}$ (O A^2) = (O A^2 + A H^2 + O H^2) ; $3\frac{1}{8}$ (D G^2) = (D G^2 + G L^2 + D L^2); and so of all the rest of the similar triangles in the diagram ; that is to say, $3\frac{1}{8}$ times the area of a square on the longer of the sides that contain the right angle = the sum of the squares of the three sides, in all these triangles.

Now, supposing us to produce the line *a b* to P, and join P H and P A, producing a rectangle O A H P, it is self-evident that the diagonals O H and P A would intersect and bisect each other at a point *n* : and it is also self-evident that, with O as centre, and O *n* as interval, we might describe a circle. I have only omitted to make these additions to avoid confusion in the diagram.

Well, then, I now beg to direct your attention to some remarkable arithmetical truths. First When O A, the radius of the circle Z, = 4, then, $\frac{\text{O H}}{2} = 2{\cdot}5 = \text{O}\ n$. Hence: We get the following curious facts. First: $2(3\frac{1}{8}\ \text{OA}^2) + 2(3\frac{1}{8}\ \text{O}\ n^2) = 2(3\frac{1}{8}\ \text{O N}^2) + 100$; or, $\{2(\text{O A}^2 + \text{A H}^2 + \text{O H}^2) + 2(3\frac{1}{8}\ \text{O}\ n^2\} = 6{\cdot}25^2 + 100$. Proof: $3\frac{1}{8}\ (\text{O A}^2) = 3{\cdot}125\ (4^2) = 3{\cdot}125 \times 16 = 50$; and, $3\frac{1}{8}\ (\text{O N}^2) = 3{\cdot}125\ (2{\cdot}5^2) = 3{\cdot}125 \times 6{\cdot}25 = 19{\cdot}53125$; therefore, $2\ (50 + 19{\cdot}53125) = 2 \times 69{\cdot}53125 = 139{\cdot}0625$. But, $6{\cdot}25^2 = 39{\cdot}0625$, therefore, $39{\cdot}0625 + 100 = 139{\cdot}0625 = \{2\,(\text{O A}^2 + \text{A H}^2 + \text{O H}^2) + 2(3\frac{1}{8}\ (\text{O}\ n^2)\} = 2(3\frac{1}{8}\ \text{O A}^2 + 2(3\frac{1}{8}\ \text{O}\ n^2)\}$. Second: When C A the diameter of the circle Z = 8, $\frac{4}{5}$ (C A) = 6·4 = C D. Now, $\frac{\text{C D}}{5} = \frac{6{\cdot}4}{5} = 1{\cdot}28$; and, $1{\cdot}28 : 1 :: 1 : \frac{3{\cdot}125}{4}$ that is, $1{\cdot}28 : 1 :: 1 : {\cdot}78125$. Hence: $\frac{{\cdot}78125}{1}$ and $\frac{1}{1{\cdot}28}$

are equivalent ratios. In previous Letters I have drawn the only legitimate inference which can be drawn from these facts; and I now put through you, my dear Sir, the following questions to Mr. R——. What can the value of π be, but 3·125? What can the true expression of the ratio of diameter to circumference in a circle be, but $\frac{1}{3 \cdot 125}$?

Now, I have proved that D G H is a commensurable right-angled triangle, of which the sides that contain the right angle, are in the ratio of 7 to 24; and I have also proved that D G A is a commensurable right-angled triangle, of which the sides that contain the right angle are in the ratio of 3 to 4. Now, the angle A D G is divided by D H, the hypothenuse of the triangle D G H, into two angles, A D H and H D G, and the angle A D H = the angle D A H. But the angle F D A = the angle D A H; therefore, the three angles H D G, A D H, and F D A, are together equal to the right angle F D G. Well, then, through you, my dear Sir, I beg to ask the following favour of Mr. R——. Will that gentleman be good enough to tell us the values of the three angles H D G, A D H, and F D A, expressed in degrees?

The following is the method of constructing the enclosed diagram, No. 2. *(See Diagram XVII.)*

Let A and B be two points dotted at random. Join A B. Post I. On A B describe the equilateral triangle O A B. (Euclid: Prop I, Book I.) Bisect the angles of the triangle and their opposite sides. (Euclid: Book I, Prop. 9 or 10). The sides of the triangle are bisected at the points C, D, and E. With O as centre, and O A or O B as interval, describe the circle X. From the points C

and D draw straight lines parallel to A E, to meet and terminate in the side of the triangle O B, bisecting E O at F, and E B at G. Produce C F to meet and terminate in the circumference of the circle X, at the point H. Produce B O to meet and terminate in the circumference of the circle X at the point K, and join K H and B H, producing the right-angled triangle K H B. (Euclid: Prop. 31, Book 3). From the angle B, in the triangle K H B, draw a straight line at right angles to K B, and therefore tangental to the circle X, to meet a perpendicular let fall from the angle H at the point P, producing the rectangle F B P H. Produce K H to meet B P produced at the point M, by which we get the right angled triangle K B M. From the point O, the centre of the circle X, draw a straight line parallel to K M, to meet and terminate in the line B P, at the point T, and so, bisecting the lines B H and B M. From the angle H, in the triangle K H B, draw a straight line through the point O, the centre of the circle X, to meet and terminate in the circumference of the circle at the point L, and join K L and B L. Produce H C to meet and terminate in the line B L at the point N, producing the right-angled triangle H B N, which is similar and equal to the right-angled triangle B H M. Join N M. Then: N M and H B are diagonals of the rhomboid N B M H, and intersect, and bisect each other, at the point *n*. With K F as diameter describe the circle Y. Join H T.*

Now, B *n* is a straight line drawn from the angle B, in

* Because K B is bisected at O, and O T parallel to K M, by construction: it follows, that B M the base of the right-angled triangle K B M, is bisected at the point T: and because F H is parallel to B M by construction, it follows that B P = F H. Hence: T P = $\frac{7}{24}$ (B T) or, $\frac{7}{24}$ (H P).

the right-angled triangle O B T, perpendicular to its opposite side O T, and O n B and B n T are similar triangles, and similar to the whole triangle O B T. Hence: O B^2 — O n^2 = B n^2; and, B T^2 — B n^2 = T n^2; and when O B = 4, O n = 3·2, and T n = 1·8. Therefore, O B^2 — O n^2 = B T^2 — T n^2; and this equation = B n^2. Proof: BT = $\frac{3}{4}$ (O B), and O T = $\frac{5}{4}$ (O B), by construction; and when O B = 4, B T = 3, and O T = 5. But, O B T and K B M are similar triangles, and K M = 2 (O T), by construction; therefore, K M = 10 when O B = 4. But, K B = $\frac{4}{5}$(K M), and B H = $\frac{3}{5}$ (K B), by construction; therefore, K B = 8, and B H = 4·8, when O B = 4. Hence, $\frac{\text{H B}}{2} = \frac{4{\cdot}8}{2}$ = 2.4 = B n: O n = 3·2: and T n = 1·8: therefore, O B^2 — O n^2 = B T^2 — T n^2; that is, $4^2 - 3{\cdot}2^2 = 3^2 - 1{\cdot}8^2$; or, 16 — 10·24 = 9 — 3·24 = 5·76 = B n^2; therefore, $\sqrt{5{\cdot}76}$ = 2·4 = $\frac{1}{2}$ B H. You will observe that the triangles O B T and K B M are partly within, and partly without, the circle X.

Now, we have in the geometrical figure represented by this remarkable diagram several right-angled triangles altogether within the circle, and in one of these triangles, K H B, H F is a straight line, perpendicular to its opposite side K B. This is self-evident. Hence: according to Euclid, Prop. 8: Book 6: H F K and H F B are similar right-angled triangles, and similar to the whole triangle K H B. Now, if the 8th Proposition of the 6th book of Euclid were of general and universal application, H K^2 — K F^2 should equal H B^2 — B F^2 and both should equal H F^2. But this is not a fact!!!

Proof: K H = $\frac{4}{5}$ (K B), and H B = $\frac{3}{5}$ (K B), and when K B = 8, K H = 6·4, and H B = 4·8. But, K F = 5, and F B = 3; therefore, H K^2 — K F^2

$= 6{\cdot}4^2 - 5^2 = 40{\cdot}96 - 25 = 15{\cdot}96$; and the square root of $15{\cdot}96 = 3{\cdot}99$, &c., &c. And, H B^2 — F $B^2 = 4{\cdot}8^2 - 3^2 = 23{\cdot}04 - 9 = 14{\cdot}04$; and the square root of $14{\cdot}04 = 3{\cdot}74$ &c., &c.; therefore, the right-angled triangles on each side of H F are not similar to the whole triangle K H B, or similar to each other.

Now, the diameter of the circle $X = 8$, and the diameter of the circle $Y = 5$. Hence: Twice the sum of the areas of the circles X and $Y = \{(2\,\pi)^2 + K\ M^2\}$. And the true arithmetical value of π is $3{\cdot}125$, whatever Mr. R—— may say.

May 26th.

I had written so far yesterday, but deferred finishing and posting the Letter, thinking I might get Mr. R——'s rejoinder to my last by this morning's post, and my conjecture proved true; your esteemed favour of yesterday is to hand. Mr. R——'s Paper commences with a mistake, and I shall not say more about it to-day; but I venture to tell you that if he reads this Letter with care, he will not be "*as the hungry man who dreams of food, and wakes, and lo! there is none.*"

Believe me, my Dear Sir, very truly yours,

JAMES SMITH.

J—— S——, Esq.

Mr. JAMES SMITH *to* Mr. R——.

BARKELEY HOUSE, SEAFORTH,
27th May, 1868.

MY DEAR SIR,

I have more than one friend that takes as deep an interest as yourself in my mathematical labours, and one of these I happened to meet yesterday, and read

to him that part of my Letter to you, in which I point out that the 8th proposition of the 6th book of Euclid is not of general application, and therefore, not true under all circumstances. He got deeply interested, and requested I would let him have the Letter, so that he could give it a careful reading, and he would take care that it was duly posted. As it contained no secrets, I left it with him. On seeing him to-day, he told me he got so interested in the perusal, that he only got it posted by taking it to the general post office, and paying for it as a late letter, and I presume you would find it so marked. This will explain how my two Letters were not posted together.*

I commence this Letter with the intention of finishing and posting it, before I can possibly have Mr. R——'s reply to mine posted yesterday. If that gentleman happen to have read my communication of the 23rd inst., with only ordinary care, he cannot fail to discover the blunders into which he has fallen in his last Paper, and this will afford him the opportunity of correcting his mistakes, before they are pointed out to him by me.

Mr. R——'s Paper commences thus:—"There is an oversight in Mr. Smith's Paper of the 19th. On page 4 he makes E B = B L, by construction. *(See Diagram X V.)* Then: since E B : E F : : B L : B O ; E F = B O. That is, E F = O F, or, the hypothenuse = one of the sides." Now, O F = O B, for they are radii of the circle X, and E F is obviously a shorter line than O F. No doubt, Mr. R——'s assertion, reasoning, and conclusion, would be true, if the 8th proposition of Euclid's 6th book were of universal application, and true under all circumstances. But this is not a fact! Hence: although F E is a straight

* The second Letter had reference to a pure matter of business.

line drawn from the angle F, in the right-angled triangle G F B, perpendicular to its opposite side G B, the triangles on each side of F E are not similar to the whole triangle G F B, and to each other; and it follows of necessity, that EB is *not* to E F as B L to B O. Mr. R—— next says:—"Mr. Smith forgot that in his Letter of the 4th, he took E, making O E = $\frac{1}{4}$ (O B), raised E F perpendicular, and drew O L parallel to G F, and G F produced to H." Mr. Smith begs to inform Mr. R——, that he *forgot* nothing in his Letter of the 4th, that he intended to say; and *did* nothing so absurd as to take E and make O E = $\frac{1}{4}$ O B. No arithmetical symbol whatever entered into Mr. Smith's description of the method of constructing the diagram No. 2, in his Letter of the 4th. Mr. Smith may be a blunder-head, but he never made so glaring a blunder as this. Mr. R—— then observes:—"This fixes B L at what I gave it, $\frac{1}{2}$ B H = $\frac{4}{5}(\sqrt{15}) = 4\,(\sqrt{\frac{3}{5}})$. The difference between us arises from the fact that Mr. Smith has forgotten what follows from his own constructions." Mr. Smith begs to inform Mr. R—— that it is *not a fact* that Mr. Smith "*has forgotten*" what follows from his own constructions. Mr. R—— then says:—"If the construction be B L = 3, and G H be drawn parallel to O L, then, B E = 2·88, not 3: and G E = 5·12." If Mr. R—— had even looked at the diagram No. 2, in my Letter of the 19th, with the *eye* of a Geometer, he would never have written such nonsense as this. O B is obviously divided into 4 equal parts, and 3 times O E = E B, and 5 times O E = G E; and Mr. R—— has only to take his compasses, and with B as centre, and B E as interval, describe an arc, and he will find that the points E and L will be the extremities of the arc. Mr. R—— then

observes:—"If, on the other hand, the construction be O L parallel to G H, the direction of G H being fixed by E F, G E being = 5, then, the values are different; they are as I gave them, if I committed no mistake." But you have committed a very great mistake, Mr. R—— ! $BL = \frac{3}{4} OB = BE$, therefore, B E and $BL = \frac{3}{8}(GB)$. Hence: $EF = (BL + LK) = (3 + {}^{7}\ (3))$ not $\frac{4}{5}(\sqrt{GE \cdot EB})$ as you put it. Mr. R—— next says:—"If $GE = 5$, $EB = 3$, $BL = 3$, and O L joined, then, O L is not parallel to G H; for, if so, B O L and B E F are similar, and $BO = EF$ as above." This is, indeed, not an *ordinary*, but a *marvellous* blunder. For, it is self-evident, that E F is parallel to B H, and therefore perpendicular to G B, and according to Euclid, Prop. 8, Book 6, the triangles, F E G and F E B, on each side of E F, are similar to the whole triangle G F B, and similar to each other. But, according to Euclid, prop. 47, book 1, $GE^2 + EF^2 = GF^2$, and $FE^2 + EB^2 = FB^2$; and it follows of necessity, that $GF^2 - GE^2 = FB^2 - BE^2$, and that this equation $= EF^2$. Where will you be, Mr. R——, if you try to prove this equation? Mr. R—— then says:—"Hence, Mr. Smith's difficulty (in page 7) to prove 5 = 5·12." Had Mr. R—— taken the hint I gave him (in page 7), he would have discovered that the difficulty was his, not mine, and that his difficulty arose from the fact of his inability to prove 5·12 = 5. This remarkable paragraph of Mr. R——'s last Paper concludes as follows:—"It is not mathematics at fault, but simply an oversight in mixing up two constructions." Well, then, it remains for Mr. R—— to get rid of two constructions, and prove that by the 8th proposition of Euclid's 6th book, and the 47th proposition of Euclid's 1st book, the triangles on each side of E F are similar to the whole triangle G F B, and similar to each other.

The second and only remaining paragraph of Mr. R——'s Paper commences thus:—"Again: pp. 4 and 5, $y : x :: 3 : 4$; and $y : z :: 9 : 16$. *(See Diagram XIV.)* He (Mr. Smith) can prove these facts by means of any hypothetical value of π intermediate between 3 and 4, *so that it be finite and determinate*, and it follows of necessity, that π cannot be indeterminate." What Mr. R—— may say to this I know not. I give him the facts, &c. "Why does Mr. Smith introduce the words, '*so that it be finite and determinate?*'" I cannot tell what Mr. R—— means by putting this question, unless he means to insinuate that *arithmetic* is not a branch of mathematics. Mr. R—— next says:—"You can prove that circles are as the squares of their radii, by any *hypothetical value of* π, whether or not it be finite and determinate." Now, π is a mere symbol, and *per se* can prove nothing. We may call $\pi = 3{\cdot}1416$, $3{\cdot}14159$, $3{\cdot}14159265$, or we may add as many more decimals as we please, and with any of these finite values of π, we may prove that the areas of circles are to each other as the squares of their radii, and the circumferences of circles to each other as their diameters; but it appears to me, that to do this, we must put an arithmetical value on π; and not only so, but we must make that value finite and determinate. Mr. R—— then observes:—"Let π = any hypothetical quantity, then, $y : x :: 9\pi : 12\pi :: 3 : 4$. The *arithmetical value of* π is *not* necessary to the proof of the ratio. Hence the conclusion does not follow from the premisses, and still waits for its much-needed proof." Now, suppose me to put it: $y : x :: 9$ (something) : 12 (something)—the something being neither a finite number, a terminating decimal, nor arithmetically expressible—and say I could

prove something by the formula. Would not this be nonsense? But further: must we not put a value on π to find the values of y and x in the first place, before we can go in search of the ratio. Hence: Mr. R——'s example of continued proportion proves nothing *per se*. So much for Mr R——'s last Paper, which appears to me to be a series of blunders from beginning to end.

Well, then, these facts put Mr. R—— on the horns of a dilemma. In Euclid, the 8th proposition of the 6th book is inconsistent with the 47th proposition of the 1st book, therefore, both cannot be true. Which is true? Mr. R—— may take his choice!

The enclosed diagram is a fac-simile of the diagram No. 2, contained in my Letter of the 23rd, with certain additions. *(See Diagram XVIII.)*

The following may be taken as the construction of the geometrical figure represented by the diagram, which is somewhat varied from the construction as given in my Letter of the 23rd inst.

Let A and B be two points dotted at random. Join AB. Euclid: Post 1. On AB describe the equilateral triangle O A B. Euclid: Prop 1, Book 1. Bisect the angles of the triangle and their opposite sides. Euclid: Book 1, Prop. 9 or 10. The sides of the triangle are bisected at the points C, D, and E. With O as centre, and O A or O B as interval, describe the circle X. From the points C and D draw straight lines parallel to A E to meet and terminate in the side of the triangle O B, bisecting E O at F, and E B at G. Produce C F to meet and terminate in the circumference of the circle at the point H. Produce B O to meet and terminate in the circumference of the circle at the point K, and join K H

and BH, producing the right-angled triangle K H B. Euclid: Prop. 31: Book 3. So far the construction is precisely the same as the construction of the diagram No. 2, enclosed in my Letter of the 23rd inst. From this point I shall vary the method of construction, and comment upon these alterations as I proceed. Well, then, from the angle H in the triangle K H B, draw a straight line through the point O, the centre of the circle X, to meet and terminate in the circumference at the point L, and join KL and BL, producing the rectangle KLBH, which is an inscribed rectangle to the circle X. Then: HL and KB are diameters of the circle, and diagonals of the rectangle, and intersect and bisect each other at the point O, the centre of the circle, just as certainly as would the diagonals of an inscribed square intersect and bisect each other at the centre of the circle. But the diagonals of the rectangle are commensurable, while the diagonals of an inscribed square to the circle would be incommensurable. Produce H C to meet and terminate in the line B L, at the point N, producing the right-angled triangle H B N. Then: H N L is an oblique-angled triangle, therefore, $HN^2 + NL^2 + 2(NL \cdot NB) = HL^2$. Euclid: Prop. 12, Book 2. From the angles H and B, in the triangle K H B, draw straight lines tangental to the circle X. These lines meet at the point T, therefore H T = B T, and H T B and H O B are isosceles triangles. Join O T. Then: B H is bisected by O T at the point *n*, therefore, O *n* H and O *n* B are similar and equal right-angled triangles, and T*n*H and T*n*B are similar and equal right-angled triangles. But in the triangles O *n* H and O *n* B, the side O *n* is common to both; therefore, $OH^2 - On^2 = OB^2 - ON^2$, and this equation $= nH^2$ or nB^2. And, in the

triangles T n H and T n B, the side T n is common to both; therefore, $TH^2 - Tn^2 = TB^2 - Tn^2$, and this equation $= nH^2$ or nB^2. Hence: the triangles H n O and H n T on each side of n H are similar to the whole triangle O H T, and similar to each other; and the triangles B n O and B n T, on each side of n B, are similar to the whole triangle O B T, and similar to each other; therefore, the triangles H n O, H n T, B n O, B n T, O H T, and O B T, are similar triangles. But H B = 4·8, and O T = K F = 5, when the diameter of the circle X = 8, and it follows of necessity, that all these triangles have the sides that contain the right-angle in the ratio of 3 to 4. Mr. R—— will not require me to go into the figuring to prove these facts. Produce B T, to meet a perpendicular let fall from the angle H in the triangle K H B, at the point P, producing the rectangle F B P H. Produce B P to meet K H produced at the point M, thus obtaining the right-angled triangle K B M. Then: the triangles B H K and B H M, on each side of B H, are similar to the whole triangle K B M, and similar to each other, and the sides that contain the right angle in these triangles are also in the ratio of 3 to 4. O T, the hypothenuse of the right-angled triangle O B T, and H N the hypothenuse of the right-angled triangle H B N, intersect each other at the point V. Join H T and V B, producing the rhomb V B T H; or in other words, producing a parallelogram of which all the sides are equal, but not all the angles. The parallelogram is divided by the diagonals H B and V T into 4 similar and equal right-angled triangles, and the sides that contain the right angle in these triangles are also in the ratio of 3 to 4. H N, the hypothenuse of the right-angled triangle

HBN, is bisected at the point V. For, VN = BT, and VT = N B, and VB is a diagonal of the rhomboid V N B T. But, V H = T M, and V T = H M, and H T is a diagonal of the rhomboid VTMH, and proves that H N, the hypothenuse of the right-angled triangle H B N, is bisected at the point V. Join N M. Then: In the rhomboid N B M H, the diagonals N M and H B intersect and bisect each other, at the point *n*; that is, at the point where the diagonals of the parallelogram V B T H intersect and bisect each other. Hence: In the parallelogram VBTH both the diagonals are commensurable; while, in the rhomboid N B M H, the longer diagonal is incommensurable, and the shorter diagonal *only* commensurable. With B as centre and B F as interval describe the arc F *m* T.

Now, Mr. R—— has told us that I can *offer him nothing in Geometry that he cannot understand;* and that *we can prove nothing by practical Geometry.* But, in my last Letter I have proved, by practical Geometry, that even Euclid is at fault, and we can prove the same fact by means of the enclosed diagram.

Now, F *m* T B is a quadrant of a circle; therefore, B T = B F. But, H V = B T; therefore, H V = B F. But, since H F = B P, and H V = B T, it follows, that V F = T P; and, that B F V and H P T are similar right-angled triangles. Hence: In the triangles B F V and H P T, the hypothenuses are equal to one of the sides. To prove this in another way. Because N H is bisected at V, and B M at T, V B H and T B H are similar and equal isosceles triangles, of which H B is the base, and common to both triangles; and, V B T and V H T are similar and equal isosceles triangles, of which V T is the base, and common to both

triangles; therefore, V B, and H T = V H and B T. But, B T = B F; therefore, H V and H T = B F, and B F V and H P T are similar right-angled triangles. Hence: It would again appear, that in these triangles, the hypothenuses are equal to one of the sides.*

How happens this? Is the 47th Proposition of the 1st Book of Euclid at fault? Is Geometry an inexact science? Are Geometry and Mathematics inconsistent with each other? Well, then, I cannot help thinking that, with the information I have given him in

* This was intended to be suggestive, but I might, and as it now appears, should, have given Mr. R—— "*a little figuring*" here. Because B T is at right angles to K B, a diameter of the circle X, and H T at right angles to L H, another diameter of the circle X; it follows, that H T = B T, and that V B T H is a parallelogram of which all the sides are equal; and the squares on the diagonals of any parallelogram are together equal to the squares on its sides. Now, when K B and L H = 8, then, H B = 4·8 and V T = 3·6 and the sides of the parallelogram V B T H = 3; therefore, $HB^2 + VT^2 = 4(3^2)$ = that is, $(4{\cdot}8^2 + 3{\cdot}6^2) = 4(3^2)$, or, $(23{\cdot}04 + 12{\cdot}96) = (4 \times 9) = 36$, and is equal to a square on B M.

Again: Two parallelograms may have equal sides, and yet be unequal in area. The sides of the rhomb V B T H = 3, when the diameter of the circle X = 8; and $2(Hn \times Vn) = 2(2{\cdot}4 \times 1{\cdot}8) = (2 \times 4{\cdot}32) = 8{\cdot}64$ = area. But, the area of a square of which the sides = 3 = $\cdot 3^2 = 9$. Hence: a square encloses a larger superficies than any other form of parallelogram of equal perimeter.

Again: The square on the diagonals of a quadrilateral, together with four times the square on a line that joins the middle points of the diagonals, is equal to the sum of the squares on the four sides of the quadrilateral. Now, O H T B is a quadrilateral, and when K B and L H = 8, then O T = 5, H B = 4·8, H T and B T = 3, and O n = 3·2; therefore, $On - \frac{OT}{2} = 3{\cdot}2 - 2{\cdot}5 = {\cdot}7$, and ·7 represents the length of the line that joins the middle points of the diagonals O T and H B; therefore, $(OT^2 + HB^2 + 4({\cdot}7^2) = 2(OB^2) + 2(BT^2)$; that is, $\{5^2 + 4{\cdot}8^2 + 4({\cdot}49)\} = \{2(4^2) + 2(3^2),\}$ or, $(25 + 23{\cdot}04 + 1{\cdot}96) = \{2(16) + 2(9)\} = 50 = 3\tfrac{1}{8}(OB^2)$ area of the circle X.

my last Letter, Mr. R—— will be able to trace these remarkable facts and anomalies to their true cause. This communication is already too long, and I shall stop and wait Mr. R——'s answer to my Letter of the 23rd inst. When I see the nature of his reply, I shall then know better what must be my next step.

Mrs. Smith joins me in best wishes to your good lady, daughter, and yourself, and we hope to hear soon that you are able to attend to business. Many thanks for your very kind Letter of the 27th.

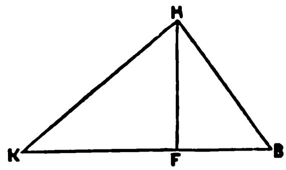
Believe me, my dear, Sir,

Very faithfully yours,

J—— S——, Esq. JAMES SMITH.

Mr. R——'s PAPER, *May 29th*, 1868.

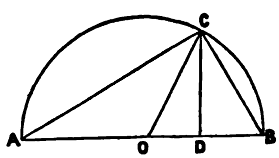
To-day I have Mr. Smith's Paper, in which he says that he disproves that the perpendicular from the right angle of a right-angled triangle divides the triangle into two that are similar to it and to one another.



Now, the construction is, K F $= \frac{5}{8}$ K B. For, F is fixed by the line C F in his figure bisecting O E. But where Mr. Smith gives his proof he says, K H $= \frac{4}{5}$ K B, H B $= \frac{3}{5}$ K B, *by construction.* But this was NOT *his construction.* If. *construction* fixes F, then K H/B H depend on it, and you cannot say K H $= \frac{4}{5}$ K B, *by construction.* You must *find* K H and B H, and certainly they are NOT $\frac{4}{5}$ K B and $\frac{3}{5}$ K B. On the other hand, if K H $= \frac{4}{5}$ K B and H B $= \frac{3}{5}$ K B (and K B $=$ 8) K F is NOT 5. This is a very remarkable oversight. The demonstration of the fallacy in Euclid is really worthy of a place in the *Athenæum*, to which Mr. Smith should send it. It may gravel De Morgan and Professor Airy.

The former part of the Paper, in which I am asked to find certain angles, is apparently a repetition of previous matter. I need not solve the problems, as I know of no other media than our ordinary tables, which Mr. Smith has found to be wrong. As to whether Mr. R—— has been caught at last, we shall see when the fact is pointed out.

That the proposition Mr. Smith impugns is universally true, is as impregnable as *any* truth can be.



A B C is *any* right-angled triangle. Bisect hypothenuse in O. Join O C.

Then, $DC^2 = CO^2 - OD^2 = AO^2 - OD^2 = (AO + OD)(AO - OD) = AD \times DB = DC^2$.

$\therefore AD : DC :: DC : DB$. Does not that prove that A D C and C D B are similar triangles?

Again, $BC^2 = CD^2 + DB^2 = AB \cdot BD = BC^2$.

$\therefore AB : BC :: BC : BD$. Does not that prove that ABC and D B C are similar triangles?

Or again, $\dfrac{AC \cdot CB}{2} = \dfrac{AB \cdot CD}{2}$ $\therefore AC : AB :: CD : CB$, and we know that angle $BAC = BCD$ always. Or again, $AC^2 = AD^2 + CD^2 = AD^2 + AD \cdot DB = AD \cdot AB = AC^2$. $\therefore AB : AC :: AC : AD$. Does not that prove A C D and A C B similar?

Again, $\dfrac{AC \cdot CD}{2}$ Sin. $ACD =$ area of triangle A D C; and angle $ACD =$ angle B; and radius : Sin. B :: BC : CD $\therefore CD = BC \cdot$ Sin. B. $\therefore \dfrac{AC \cdot CD}{2}$ Sin. $ACD = \dfrac{AD \cdot DC}{2}$ or, $AC \cdot CD \cdot$ Sin. B $= AD \cdot BC \cdot$ Sin. B.

$\therefore AC . CD = AD \cdot BC$. $\therefore AD : DC :: AC : BC$. Does not this also prove A B C and A D C to be similar?

There must be some great mistake in Mr. Smith's Paper, as pointed out above.

MR. JAMES SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH,
3rd June, 1868.

MY DEAR SIR,

Accept my thanks for your favour of yesterday, enclosing Mr. R——'s Paper. I am glad to find you are so far better as to be able to attend a little to business, and hope soon to hear of your having quite recovered.

It would indeed be folly to attempt to carry controversy further with Mr. R——; but whatever that gentleman may say, I venture to tell you, that I have proved in my last two or three letters, that the 8th Proposition of the 6th Book of Euclid is not of general and universal application, and therefore not true under all circumstances, and consequently is not only inconsistent with the 47th Proposition of the 1st Book, but also with the 12th Proposition of the 2nd Book.

I have made up my mind to bring the subject before the Public in a Pamphlet. It will take the form of a Letter to the President elect of "The British Association for the Advancement of Science." I may trouble you with another letter or two before I finish it, but in the mean time you will get a little peace.

Personally, I may say, my pride is rather gratified than wounded by the result of our correspondence, for I know that Mr. R—— is grievously mistaken. "*Magna est veritas et prevalebit.*"

Many thanks for the interest you have taken in our

discussion, and the trouble we have given you. Mrs. Smith joins me in kind regards to Mrs. S——, self, and Miss S——.

Believe me, my dear Sir,

Ever yours faithfully,

JAMES SMITH.

J—— S——, Esq.

Mr. R——'s PAPER, *June 2nd*, 1868.

In his Paper of May 4th, Mr. Smith drew O L parallel to G H, the point H having been fixed by taking O E = $\frac{1}{4}$ O B (not $\frac{1}{2}$ O B, a *lapsus* of mine). In his Paper of the 19th, he proceeds to show that Euclid is wrong; and having *repeated the above construction*, he begins by saying B E = B L, by construction. Now, I repeat that the construction which Mr. Smith reasons upon is different from the one he began with. In the construction of *4th May*, of which the succeeding diagrams are fac-similes, he fixes E by making O E = $\frac{1}{4}$ O B, and raising E H perpendicular to O B. But he *reasons* from a totally *different construction*. If B L be made = 3, or B M = 6, then the points H and E are thrown loose, so to speak, and are determined by the line B M = 6, or B L = 3. In this latter case, O E is not = $\frac{1}{4}$ O B, or B E is not = 3, it is less than 3. G E in this case = 5·12. Surely, this must be plain to Mr. Smith *now*. Euclid is all right, and Mr. Smith is all wrong, through thus mixing up these two incompatible constructions. According to his request, I, "*the precocious urchin*," found a series of ratios, the one proper to the construction he had given: B E = 3. He gave me another and different series, but these ratios were those due to the other construction, B L = B E = 3, and hence the confusion.

The same remarks apply to the *seemingly* different but *really* same construction in the Paper of the 27th. There B T and H T are drawn at right angles to O B and O H. If O T be drawn, it is parallel to K M, so that we thus have the same construction as before. But what surprises me is, that *still* Mr. Smith, after having made this diagram so often, and written so much upon it in his Papers of the 19th, 23rd, and 27th, should have overlooked the very extraordinary confusion to which I have already called his attention. In all these Papers there is this unaccountable oversight.

In this Paper of the 27th, Mr. Smith cannot proceed to say "B T = B F." It is not so; neither is B H = 4·8, nor O T = 5. A little figuring here is desirable. Is not $FH^2 = FK \cdot FB$? 15 : $FH = \sqrt{15}$. This is plain from that proposition in Euclid where it is proved that if two lines cut each other in a circle, the rectangles under the parts are equal, and that if one of them pass through the centre, cutting the other at right angles, it bisects it. Are these two propositions also untrue. If true, they show that $FH = \sqrt{15}$, and $HB = \sqrt{15 + 9} = \sqrt{24} = 2\sqrt{6}$. This is from the 47th of the 1st Book. But in such a case as this, it too may be wrong; perhaps it is not *universally* true.

Is not $KF : KB :: FH : BM$; *i.e.* $5 : 8 :: \sqrt{15} : \frac{8}{5}\sqrt{15} = BM$? Is not $KH = \sqrt{25 + 15} = 2\sqrt{10}$?

Is not $BT = \frac{4}{5}\sqrt{15}$, and $OT = \sqrt{16 + \frac{16}{25} \cdot 15} = 4\sqrt{1 + \frac{3}{5}} = 4\sqrt{\frac{8}{5}} = \frac{8}{5}\sqrt{10}$? &c.

These things cannot be gainsaid. They do not depend on similarity of the three triangles, which Mr. Smith questions. But the results which would follow from the application of the proposition which he calls in question would be the same as these. For instance, $KF : FH :: KH : HB$; *i.e.* $5 : \sqrt{15} :: 2\sqrt{10} : \frac{2}{5}\sqrt{150}$. $\therefore BH = 2\sqrt{\frac{150}{25}} = 2\sqrt{6}$, the same as above. And so $KH : HB :: KB : BM$; *i.e.* $2\sqrt{10} : 2\sqrt{6} :: 8 : 8\sqrt{\frac{6}{10}} = 8\sqrt{\frac{3}{5}} = \frac{8}{5}\sqrt{15} = BM$, same as above.

And HF : HB :: OB : BT; *i.e.* $\sqrt{15} : 2\sqrt{6} :: 4 : 8\sqrt{\tfrac{6}{15}} = \tfrac{8}{5}\sqrt{10} =$ BT, as before; and so on. I hope Mr. Smith now sees his mistake, and the absolute and universal truth of Euclid.

Mr. J—— S—— *to* Mr. James Smith.

4th June, 1868.

My Dear Sir,

I beg to thank you for your kind note of yesterday, and now send you another *philippic* from my relative, Mr. R——, who, you will see, upholds Euclid very determinedly. All I would say is—"With good advice make war." Consult some of your Liverpool Mathematicians on the subject of your Windermere discoveries, before venturing into print. Your doing so will, all the better, prepare you for what you contemplate, and can do no harm. It is so long since *I* grappled with these studies, that I don't venture *now* to meddle with them, especially in my present state of health. The time was, when I would have entertained them with no ordinary relish.

Thanking you for your kind wishes, and glad to say I go on improving, I hope, I remain, with Mrs. S—— and my daughter's very respectful regards to all at Barkeley House,

My dear Sir, yours faithfully,

J—— S——.

James Smith, Esq.,
Barkeley House.

MR. JAMES SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH,
6th June, 1868.

MY DEAR SIR,

Your favour of the 4th only came to hand yesterday afternoon, too late to reply by our last post from Seaforth. I have carefully gone through Mr. R——'s Paper of the 2nd June, which concludes by his observing: "*I hope Mr. Smith now sees his mistake, and the absolute and universal truth of Euclid.*" This Paper does not convict, and therefore cannot convince me, of any mistake, or of "*the absolute and universal truth of Euclid.*" It has made me more certain than ever—if possible—that I am right; but, to continue our correspondence with any chance of Mr. R—— and me arriving at "*a happy state of concord*" on the value of π, and the ratio of diameter to circumference in a circle, is perfectly hopeless. I shall therefore abide by my resolution, and close the controversy.

I thank you, my dear Sir, for your well-intentioned advice; but, "*the die is cast,*" I am in print, and it will now be for others to judge between us. As Mr. R—— is so very confident, it may be that, like my correspondent Mr. Gibbons, he may have no objection, and might even wish to see his name in print in connection with our controversy; if so, I shall be happy to gratify him.

Had it not been for my discoveries at Windermere, I should have dealt with Mr. R—— in a different way. In my Letter of the 21st April, I gave him a theorem for solution, arising out of a geometrical figure represented by a diagram

enclosed in that letter. *(See Diagram IX.)* In his Paper, without date, which came into my hands on the 8th May, and therefore must have been written about the 4th, Mr. R—— solves the theorem after a certain fashion, finds the number of times the area of the circle P is contained in the area of the circle X Z, making it, quite correctly, to be 15·2587890625, and he then jumps to the following conclusion: "*The values of* O R, R V *and* O V, *given here correspond to* O K = 2, *not to* $\pi \times OK^2 = 60$." I passed this by at the time, on the principle that the lesser must give way to the greater; and, knowing that my proofs of *Euclid at fault* are irrefragable, I certainly thought I should have been able to convince Mr. R—— of this fact, and consequently, that our agreement on minor points would necessarily follow.

Now, my dear Sir, it will be obvious to you, that it would be very absurd if I were to say, that under no conceivable circumstances can the area of the circle P = 60. *(See Diagram IX.)* Now, since the area of the circle P is contained 15·2587890625 times in the area of circle X Z,—and on this point Mr. R—— and I are agreed—it follows of necessity, that if the area of the circle P = 60, 60 × 15·2587890625 = 915·52734375 = area of the circle X Z; and, since $\pi r^2 =$ area in every circle, it follows that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle.

Now, suppose me to have given Mr. R—— the following theorem for solution.

Let the area of the circle X Z = 915·52734375. Find the radius of the circle X Z, and the area of the circle P.

Where would Mr. R—— have been? How would he

have gone to work with an indeterminate value of π? Could he have found the area of the circle P, unless he had known how many times P is contained in X Z? Did it ever occur to Mr. R—— to construct the geometrical figure represented by the diagram, and so discover that the area of P is contained 15·2587890625 times in the area of X Z? Let him think of this, and then go to work to demonstrate his oft repeated assertion, that "*practical geometry can prove nothing.*" Now, it is impossible that Mr. R—— can dispute that the area of the circle X Z = 915·52734375 when the area of the circle P = 60, and he has admitted that $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle. Well, then, let Mr. R—— try to find $\sqrt{\left(\frac{915·52734375}{\pi}\right)}$, and where will he be, if he go to work with any other value of π but that which makes 8 circumferences = 25 diameters in every circle? Mr. R—— *might* say, that $\sqrt{\left(\frac{915·52734375}{3·2}\right)}$, $\sqrt{(286·102294921875,)}$ and then tell me that 3·2 is as good a value of π as 3·125. No doubt for certain purposes it is, and many things in Mathematics can be proved as well by the one as the other. But, let Mr. R—— take π = 3·2, find the radii of all the circles, and prove that the area of the circle P = 60, and where will he be?

In my Letter of the 24th April, I solved the theorem given to Mr. R—— for solution, in my Letter of the 21st. Miss S—— told me, on the 27th April, that that communication had been forwarded to Mr. R—— on the 25th—I intended it to have been *entre nous* for a time—so that he was actually in

possession of my solution of the theorem when he wrote the Paper which gives me his; but of this fact he has never made mention.*

We are glad to hear you go on improving, and with our united kind regards to all,

Believe me, my dear Sir,

Ever yours most truly,

JAMES SMITH.

J—— S——, Esq.

* The solution of the theorem given in my Letter of the 21st April, not only required Mr. R—— to find the number of times the area of the circle P is contained in the area of the circle X Z; but also to find the values of the sides of the right-angled triangle O V B, when the area of the circle P = 60 .This he passed over, *sub silentio.*

Let the radius of the circle P, in the diagram No. IX, (page 135) = 2.

Then:

$3\frac{1}{8}(2^2) = 12\cdot5$ = area of the circle P.

$3\frac{1}{8}(4^2) = 50$ = area of the circle X.

$3\frac{1}{8}(5^2) = 78\cdot125$ = area of the circle Y.

$3\frac{1}{8}(6\cdot25^2) = 122\cdot0703125$ = area of the circle M.

$3\frac{1}{8}(7\cdot8125^2) = 190\cdot73486328125$ = area of the circle X Z.

In the circles Y, M, and X Z, the radius is the hypothenuse of the right-angled triangles O B C, O C F, and O F R: and in the right-angled triangles O B C, O C F, O F R, and O R V, the sides that contain the right angled are in the ratio of 3 to 4, by construction; and it follows, that the ratio of perpendicular to hypothenuse in all these triangles is as 4 to 5; therefore, $\frac{5}{4}$ (O R =) $\frac{5 \times 7\cdot8125}{4} = \frac{39\cdot0625}{4} = 9\cdot765625$ = O V, the hypothenuse of the right-angled triangle O R V. Hence: O V $= 3\cdot125^2$.

Now, it is self-evident, that with O as centre, and O V as radius, we might describe another circle (X X Z). Conceiving this addition to have been made to the diagram, it follows, that $3\frac{1}{8}(9\cdot765625^2) = (3\cdot125)^5$ and that this equation or identity $= 298\cdot0232238769531 25$ = area of the circle X Y Z: it therefore follows, that the area of the

MR. JAMES SMITH *to* MR. S——.

BARKELEY HOUSE, SEAFORTH,
8th June, 1868.

MY DEAR SIR,

I found your favour of yesterday awaiting me on my return from Liverpool to-day, at 6 p.m., which only gives me half-an-hour to reply to save this day's post.

circle P is contained 4 times in the area of the circle X: 6·25 times in the area of the circle Y: 9·765625 times in the area of the circle M: 15·2587890625 times in the area of the circle XZ: and, 23·841857910 15625 times in the area of the circle XYZ. Hence: $\frac{3\frac{1}{8}(OV^2)}{3\frac{1}{8}(OK^2)} = \frac{OV^2}{OK^2}$. But, $\frac{298·0232238769 53125}{12·5} = 23·8418579101 5625$: and this equation or identity, is equal to the number of times the area of the circle P is contained in the area of the circle XYZ: and it follows, that the arithmetical value of π—or in other words, the circumference of a circle of diameter unity, or, the ratio of diameter to circumference in a circle—is inseperably connected with the multiples of 2 and 5.

In his Paper of the 4th May, Mr. R—— makes the following assertion. "*The values of* O R, R V, *and* O V *given here correspond to* O K = 2, *not to* π (O K²) = 60." (See page 125). The foregoing facts resolve this *assertion into a gross absurdity*. For, it may be proved, that we necessarily arrive at the same result, whether we make the computations from a given value of the diameter, circumference, or area of the circle P. Let the "reasoning geometrical investigator," and honest "recognised mathematician," who is a sincere and earnest enquirer after scientific truth,—if such an one is to be found—work out the computations.

Again: When radius of the circle P = 2, the area of the circle = 12·5, and the area of the circle XYZ = 298·023223876953125: and it follows, that $\frac{298·023223876953125}{12·5} = 23·841857910 15625$ is the number of times the area of the circle P is contained in the area of the circle XYZ. Now, 12 times 23·84185791015625 = 286·102294921875

In a correspondence I had with an eminent mathematician in 1860–1861, which I published, I gave him the opportunity—and he embraced it— of revising his own Letters; and I shall take care that your relative has the opportunity of revising his Papers; but, I am sure you have too much good sense to suppose, that having made such an important discovery as *Euclid at Fault,* (and your relative's Papers prove it,) I should let the discovery die with me. No! No! This would be a moral crime. I know my duty, and must perform it, even if it should be at the expense of giving pain to others, which I should much regret. Truth, however, admits of no compromise, and it would be a dereliction of duty, if, knowing a truth of the utmost importance in the interests of mankind, I hesitated to proclaim it. Be assured, however, that so far as I can consistently do so, I shall strictly comply with your suggestions. With kind regards,

Believe me, my dear Sir,

Very truly yours,

J—— S——, Esq. JAMES SMITH.

and 286·102294921875 : 298·023223876953125 : : 3 : 3·125 : and it follows, that 286·102294921875 is the area of a regular inscribed dodecagon to a circle of which the area is 298·023223876953125.

Proof: $\sqrt{\frac{\text{area}}{\pi}}$ = radius in every circle; and 6 (radiux × simi-radius) = area of a regular inscribed dodecagon to every circle. Now, $\sqrt{\left(\frac{298{\cdot}023223876054125}{3{\cdot}125}\right)} = \sqrt{(95{\cdot}367431640625)}$ = radius of circle X Y Z : $\frac{1}{4}(193{\cdot}367431690625) = \sqrt{(23{\cdot}841857910015625)}$ = semi-radius of the circle X Y Z; therefore, 6 (radius × semi-radius) = $6(\sqrt{95{\cdot}367431164625} \times \sqrt{23{\cdot}841857910015625}) = 47{\cdot}68371582003125$, and it follows, that $6(47{\cdot}68371582003125) = 286{\cdot}102294921875$ = area of a regular inscribed dodecagon to a circle of which the area = 298·023223876953125. Surely it cannot be necessary to carry argument further, with any honest "recognised Mathematician."

BARKELEY HOUSE, SEAFORTH,
6th July, 1868.

MY DEAR SIR,

On the 1st inst. I received, through my Printers, Messrs. A. & D. Russell, an extraordinary "*philippic*," from your relative Mr. R——, of which the following is a copy:

30th June, 1868.

SIRS,

I have your Letter, conveying Mr. Smith's refusal to allow me to correct my notes to my own mind. I *erased nothing* that affected the *argument or the question under discussion*. The *italics* are simply ridiculous; had I prepared these notes for the press there would hardly have been one italic in the whole series. Mr. Smith never asked my leave to make this use of these notes: and I consider that he does me an injustice, in not only publishing them, but publishing them in the rude and imperfect manner in which they were hurriedly written. I insist on the erasure of all references to individuals, of such a kind as to offend them.

There are references to Professor de Morgan, which Mr. Smith even cannot say affect *his* Letter, though they may gratify his spleen against that gentleman.

I see in the proof sent me, to-day, a Letter of Mr. S——'s, which I am sure that gentleman would refuse to see in print. What kind of connection has *it* with the discussion? But it contains a reference to the *Athenæum*! This kind of thing is not honourable. If I express my mind freely to a gentleman about another, the former has no right to repeat my words, to express his own feelings against the latter.

If I am not to be allowed to make my own notes

decent, of what use is it to send the proofs to me? I have not read the new proof; and desire Mr. Smith to know that I consider him making an unfair and vexatious use of these notes.

The only objection, I can imagine to my corrections, is the expense of them, but that is not my fault.

R——.

Mr. R—— returned what he calls the "*new proof*," and the manuscript of it, and with them came this "*extraordinary philippic.*" When he wrote it he must not only have lost his temper, but his memory. You will observe that he says:—"*I see in the proof sent me to-day a letter of Mr. S——'s*," (this had reference to his Paper of the 23rd April,) and if I understand written language, he means to say, that this document came into his hands for the first time on the 30th June, the date of his "*philippic.*" Now, what are the facts? When Mr. R—— returned the revised proofs of his Papers of the 2nd, 8th, and 16th April, he had erased not only sentence after sentence, but even whole paragraphs, and in such a fashion, as to resolve my replies into "*perfect nonsense.*" This, of course, it was impossible I could submit to, and I instructed my Printers to throw off fresh proofs in the form in which I thought they should appear. In revising I omitted everything which Mr. R—— had erased, so far as I could do so in justice to myself, and even left out some sentences much to my own mortification. These revised proofs were sent to Mr. R—— on the 26th June, and with them the first proof of his Paper of the 23rd April, which he calls the "*new proof.*" None of these proofs being returned, and waiting to print off, I instructed Messrs.

Russell, on the 30th June, to send the following telegram. "*Won't print off before to-morrow, please return proofs by to-night's post.*" This brought the "*new proof*" unrevised, and with it came Mr. R——'s "*extraordinary philippic.*" "*Those who live in glass houses, should not throw stones.*"

Well, then, Mr. R——'s Papers of the 19th and 26th February, 7th, 20th and 27th March, and 1st April, have been revised by his own hand, from his own manuscripts, and are now in print. With one exception, they appear exactly as the revised proofs were returned. The exception is the Paper of the 19th February, and the explanation is this:—My Printers, Messrs. Russell, deferred printing off for a week, giving Mr. R—— ample time to revise and return the proof. Not receiving it, they thought it was not his intention to revise, and printed off—and this was done during my absense from home. When they received the revised proof they found Mr. R—— had not altered a word, merely putting in common type most of the words that were underlined in the original manuscript. Hence, the Paper appears as it was originally written, but there is not a word altered in the text.

I enclose you strips of two Letters of yours, which will appear in the work. You will remember that in sending me Mr. R——'s Papers, your own remarks were usually very short, and generally given inside the envelope enclosing the Papers. Fron time to time I noticed your remarks in my replies. The enclosed Letters of yours in no way affect the controversy, and simply prove that at the time of writing them, I had not in your opinion written anything "*unbefitting a Christian and a gentleman.*" Now, my dear Sir, what objection can you take, and what

objection has Mr. R—— any right to take, to my inserting these two Letters? I hardly think Professor de Morgan will thank Mr. R—— for his efforts to force upon me the conviction that he (Mr. de M——) is a gentleman. This is more than the Professor would expect of me, after his Philippic in the *Athenæum* of the 28th September, 1867.

Now, my dear Sir, what is my position? According to Mr. R——'s *ipse dixit*, Professor de Morgan and the Editor of the *Athenæum*, are to be at liberty to call me all kinds of "*hard names*," and I am not a gentleman, if I adopt the only way open to me, to say "*a word for the defendant*." There is not a Scientific Journal in the United Kingdom would insert a communication of mine, now that the *Correspondent* is defunct, and how am I to say "*a word for the defendant*," except by publishing? It is not for the love of *pelf* I publish, for I can assure you publishing on mathematical subjects—so far as my experience goes—is a very improfitable speculation. In my own day, my works may probably not be read by any living mathematican, with a view to the advancement of Scientific truth. I am certain Mr. R—— has never read my Letter to the Duke of Buccleuch through. He has glanced at it, and attempted to *pick holes* where there are no *holes*. In none of his Papers has he ever referred to the proofs I have given in that Letter, that the *natural*, *geometrical*, and *trigonometrical* sine of an angle—which mathematicians maintain to be one and the same thing—may be all different. I venture to tell you, my dear Sir, that had Mr. R—— read that Letter through with ordinary care, he would never have fallen into the *egregious blunders* of which almost every one of his Papers affords the clearest evidence.

What can have put Mr. R—— so dreadfully out of temper? His Papers prove distinctly enough, that in our controversy he considered all the *sense* on his side, and all the *nonsense* on mine. If so, what can he have to fear from my publishing his Papers? If he is right, should not I be the party "*gibbeted*," not he? Since he declines to revise fairly, and now refuses to revise at all, of course his remaining Papers will appear in their original form.

In conclusion I may observe: Having been privileged to make the discovery—a discovery that has baffled the wisdom of mathematicians for upwards of 2000 years—that Euclid is at fault in one of his most important theorems, I am sure you would not wish, or expect me, to let the discovery die with me, for the purpose of sparing the feelings of one individual. God willing, in about another month I shall be able to say with Kepler, who, on making one of his great discoveries is reported to have exclaimed:—"*Nothing holds me, I will indulge my sacred fury! If you forgive me, I rejoice: if you are angry, I can bear it. The die is cast. The book is written, to be read either now or by posterity, I care not which. It may wait a century for a reader, since God has waited six thousand years for an observer!*" Were this not a quotation, I should slightly alter the last sentence, and make it more in accordance with my own religious views and feelings.

Hoping you are quite recovered from your recent attack, and with kind regards to you and your family circle,

Believe me, my dear Sir,

Very faithfully yours,

J—— S——, Esq. JAMES SMITH.

BARKELEY HOUSE, SEAFORTH,
6th July, 1868.

MY DEAR SIR,

I was up very early this morning, and had written the enclosed before breakfast. The early post brought me your favour of Saturday, and I am glad to hear you have reaped benefit from your visit to the Hydropathic Establishment, Bridge of Allan.

Your brother's diagram is really unsound. I have given him the reasons in the two or three Letters I have written him direct, and with my reasons he has never attempted to grapple. I should merely have to go over the old arguments, although I might put them in a different form. But this appears to me needless trouble, and he has never replied to my Letters.

I must say a word or two bearing upon the enclosed.

I am sure when you brought me into controversy with your relative, you did so from the kindest motives, and I am sure you will believe me when I say, I had no idea till very lately of publishing the correspondence, and I will tell you how I was led into it. When I had perfectly satisfied myself of the fact, that *Euclid is at fault* in his Theorem, Prop. 8 : book 6, my first intention was to bring it under the notice of the British Association in the form of a pamphlet, addressed to the President elect. I commenced to write the pamphlet, but found I could not manage it satisfactorily. Another idea occured to me, which would have involved slightly the controversy between me and your relative, but I found in this way I should probably do Mr. R—— an injustice, and that he

might fairly complain. I had part of this in type. Finding that *Euclid at fault*, the ratio of diameter to circumference in a circle, and the fallacious character of our mathematical tables, are so blended together, that it appeared to me there was nothing for it, but to write a long treatise, or publish our correspondence. I was not prepared for the former, and the latter appeared the only course left me. I had no right to suppose that Mr. R—— was not writing deliberately, or that he had anything to retract in what he had written, and so far as his first six Papers are concerned, he revised them without altering a word, simply putting in common type, words that were underlined in the original. Having returned the original manuscripts of Mr. R——'s Papers of 2nd, 8th, and 16th April, and on sending him second proofs, having also returned him his first revise which he has kept, I am unable to mark the sentences and paragraphs he wished erased, and so shew you how unfairly Mr. R—— wanted to deal with me. Had he really objected to my publishing, and refused to revise, in the first instance, I think it is more than probable I should have contrived to work out my original idea. It is now too late, and so far as posterity is concerned, the work will come out in the best form I can put, and you may rest assured I shall not tamper with any of Mr. R——'s Papers; and I may tell you that, I shall not make any important alterations in my Letters. Where I have blundered, the world will know it.

I think Mr. R—— may make his mind at ease. The work will not be readable to the multitude, and our Mathematical *savans* will probably not think it worth purchasing; and so, we are neither of us very likely to

be much troubled from outsiders. Professor de Morgan will of course have another poke at me.

Hoping I have done nothing that can tend to make me forfeit your respect, and with best wishes, in haste,

I remain, my dear Sir,

Yours most truly,

JAMES SMITH.

J—— S——, Esq.

Mr. J—— S—— *to* Mr. JAMES SMITH,

7th July, 1868.

MY DEAR SIR,

Your favours of yesterday, with their enclosures, have just come to hand, and been perused by me.

When you first proposed publishing, I felt convinced, and said, that my relative, Mr. R——, would take exceptions to *any papers of his appearing in print*, without his consent being given to such a proceeding, and an opportunity afforded him of revising same.

That anything *I* had said on the spur of the moment, without the most distant expectation that my effusions would go farther than your own family, would ever reach the public in any form I certainly never even imagined, more especially where names or editors were concerned.

Allow me, then, to say that I should be *greatly obliged* by your withdrawing the two Letters of 2nd and 7th April, altogether, and, indeed, *any communication of mine*, which should have been regarded solely as they were intended to be, off-hand effusions from one friend to another. On *the question* I entered not.

Permit me, further, to say, that knowing my relative, Mr.

R—— was as a student, conversant with such subjects, I placed your Papers in his hands, with no expectation that more would come out of my doing so, than his giving his opinion as to your having actually discovered the grand problem to which they more immediately refer. Placed as I am, therefore, I would be still further obliged were you yet to resume your *first intention*, of giving your Letter on "Euclid at Fault," without reference at all to Mr. R——'s correspondence. This would please Mr. R—— and me greatly, while it would sufficiently serve your purpose.

I will, I expect, meet Mr. R—— on Thursday, when I will not fail to tell him what I have suggested to you. I am not without my apprehension, that some of my off-hand effusions, in these abrupt epistles of mine, will not be very agreeable to my relative, though not intended at all by me to give occasion for any such feeling on his part, even through falling into his hands, which, of course, I never once contemplated. All I said to you, I need hardly remark, was said in the confidence of friendship.

You will excuse so much sensitiveness, both on Mr. R——'s part and mine.

Feeling still in an improving way here, and with thanks for your good wishes,

I remain, my dear Sir,

Yours faithfully,

J—— S——

BARKELEY HOUSE, SEAFORTH,
8th July, 1868.

MY DEAR SIR,

I am in recept of your favour of yesterday, which I confess has somewhat surprised me, after your very friendly note of the 4th inst., in which you make

reference to Mr. R—— going over some of my proof sheets. Had that gentleman the objection he now appears to have to my publishing, he should have raised his objection before it was too late. He should have refused to revise any of his Papers, and thus have thrown the responsibility of publishing altogether on my shoulders. But after having revised the proofs of six of his Papers, and returned them without altering a word of the text, you can hardly expect me to abandon my work and commence *de novo*, now that I have upwards of 200 pages in type.

Well, then, I shall certainly finish the work, to be read by posterity some day; but I may probably now take my own time about it, and it may not come before the public in your day (but to this I shall not pledge myself), and thus the work may be made more perfect, than if brought out too hurriedly.

So far as the next meeting of the "British Association" is concerned, I think I can secure my immediate object in another way, and, so far, comply with your wishes. With kind regards,

Believe me, my dear Sir,

Yours most truly,

JAMES SMITH.

APPENDIX.

EUCLID AT FAULT.

BARKELEY HOUSE, SEAFORTH.
11th July, 1868.

To JOSEPH DALTON HOOKER, ESQ., F.R.S., D.C.L., &c.

SIR,

I am a very old Life Member of "*The British Association for the Advancement of Science*," and have reason to believe that I am better known, than respected, by the leading Members of the Mathematical and Physical Section. The Astronomer Royal, in his opening address to that Section, at the thirty-first Meeting of the Association, held in Manchester, in 1861, observed :—"*It was known to those present that great ingenuity had been employed upon certain abstract propositions of Mathematics which had been rejected by the learned in all ages, such as finding the length of the circle, and perpetual motion. In the best academies of Europe, it was established as a rule that subjects of that kind should not be admitted, and it*

was desirable that such communications should not be made to that Section, as they were a mere loss of time."* These remarks rose out of a small pamphlet I distributed among the mathematical Members at that Meeting, a copy of which I had taken care to put the Astronomer Royal in possession of, previous to giving his opening address.

Notwithstanding the rules adopted "in the best academies of Europe," men—call them learned or call them unlearned—have not been prevented from *spending* their time—*wasting* it the Astronomer Royal would say —on such subjects as "*The Quadrature and Rectification of the Circle*;" and I am not ashamed to confess that I am among the number. My labours have led to the discovery of the remarkable fact, that *Euclid is at fault* in one of his most important theorems; that is to say, that the eighth proposition of the sixth book of Euclid is not of general and universal application, and is therefore *not true* under all circumstances; and consequently, is inconsistent with the forty-seventh proposition of the first book. The proof of this fact is so plain and simple, as to be within the capacity of any man possessed of the most moderate geometrical and mathematical attainments; nay, I might say, within the capacity of any first class school-boy; and to you, Sir, my demonstration will be as palpable, as that the square of 3 is 9.

I must now tell you how I was led to make this important discovery. I have for years attended regularly the Meetings of the British Association, but, not being allowed—by the rules of that body—to read a Paper in the Mathematical and Physical Section on "*The*

* Transactions of the British Association for 1861. Notices and Abstracts of Miscellaneous Communicatrons to the Sections. Page 2.

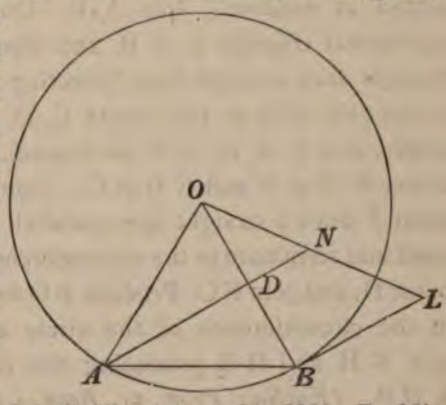
Quadrature of the Circle," I have, from time to time brought out pamphlets on the subject, and these I have freely distributed among the mathematical Members of the Association. At the last Meeting, in Dundee, I distributed one. At the time I was writing that Pamphlet, a Mr. and Mrs. S——, from Dumfriesshire, were on a visit to their son, resident in Liverpool; and being old friends of my wife's family, came out to Seaforth, and made a call upon us. I happened to mention the fact to Mr. S——, that I was engaged in writing a Letter to His Grace the Duke of Buccleuch, the President Elect of the British Association, and told him the subject of it; when Mr S—— informed me that in early life, he had himself been a good Mathematician, and still took a deep interest in Mathematics. This led us into a conversation on the subject of *squaring the Circle*, which resulted in my presenting him with several of my pamphlets. He then told me that his brother was an excellent Mathematician, and a man of leisure; and, that he had a relative residing in the immediate neighbourhood of his brother, who was a first-class Mathematician, and that he should send the pamphlets to them, and induce those gentlemen to give them their careful attention, which his own health and business engagements would not admit of *his* doing. As soon as my Letter to the Duke of Buccleuch was published, I sent Mr. S—— copies, and in December last I received a communication from him, which led to a correspondence that would make a large volume, in which his relative, whom I may call Mr. R——, played the part of my chief opponent. Only some two or three communications passed between me and my friend's brother. Mr. S—— played the part, as

it were, of a medium, or I might say referee; that is to say, my Letters were addressed to him, and after perusal, forwarded to his relative; and Mr. R——'s communications came to me through my friend, who really acted as referee, inasmuch as he kept Mr. R—— and me within the legitimate bounds of controversy.

In the course of the correspondence, I think I extracted from Mr. R—— every conceivable objection he could advance against the truth of the *theory*, that 8 circumferences $= 25$ diameters in every circle; which makes $\frac{25}{8} = 3{\cdot}125$ the true arithmetical value of π; and, $\frac{1}{3{\cdot}125}$ the true expression of the ratio between diameter and circumference, in every circle. I pointed out to Mr. R—— that in attempting to find the value of π, by multilateral-sided inscribed polygons to a circle, whether we make an inscribed equilateral triangle to the circle, or an inscribed square to the circle, our starting point, the ratio of chord to arc in every successive polygon, is a varying ratio; and I shewed him that the reason is plain enough. It follows from the fact, that the sides of every successive polygon are convergent and divergent lines from the sides of those that precede them; and consequently, that we can never, by these processes, arrive at the value of π, or the true ratio of diameter to circumference in a circle: nor can we arrive at them by any other process, in which we attempt (directly) to measure a curved line by means of straight lines. Hence, the inapplicability of the forty-seventh proposition of the first book of Euclid, to measure directly the circumference of a circle. I answered every objection started by Mr. R——, still he was not convinced; and it then occurred to me that nothing short of proving the forty-seventh

proposition of the first book inconsistent with some other theorem of Euclid, would ever convince a recognised Mathematician that the arithmetical value of π is a finite and determinate quantity. But, who ever thought of questioning Euclid? Professor de Morgan never went further than attempt to prove Euclid illogical.* But, it never entered into *his mathematical philosophy* to dream of proving Euclid *positively* at fault. How then was it likely that I should ever think of doing so?

Towards the end of April I was called away to Scotland, and on my return home spent a few days at Windermere, and it was during my stay there that I made the important discovery, that *Euclid is at fault.* The morning of the 2nd May was very wet at Windermere, and it occurred to me—as I could not leave the Hotel—that I could not better pass the time than by writing a Letter to Mr. S——, enclosing a diagram, represented by the geometrical figure in the margin, in which the angle A and the sides O B and O L in the triangles, O A B and O B L are bisected by the line A N. This I intended as introductory to a suc-



* Notes and Queries, 3rd S. VI. August 27, 1864. P. 161. Had the learned Professor asserted that the 18th and 19th Propositions of Euclid's third book are superfluous — what is proved by these propositions being established by the 16th proposition—I should have agreed with him.

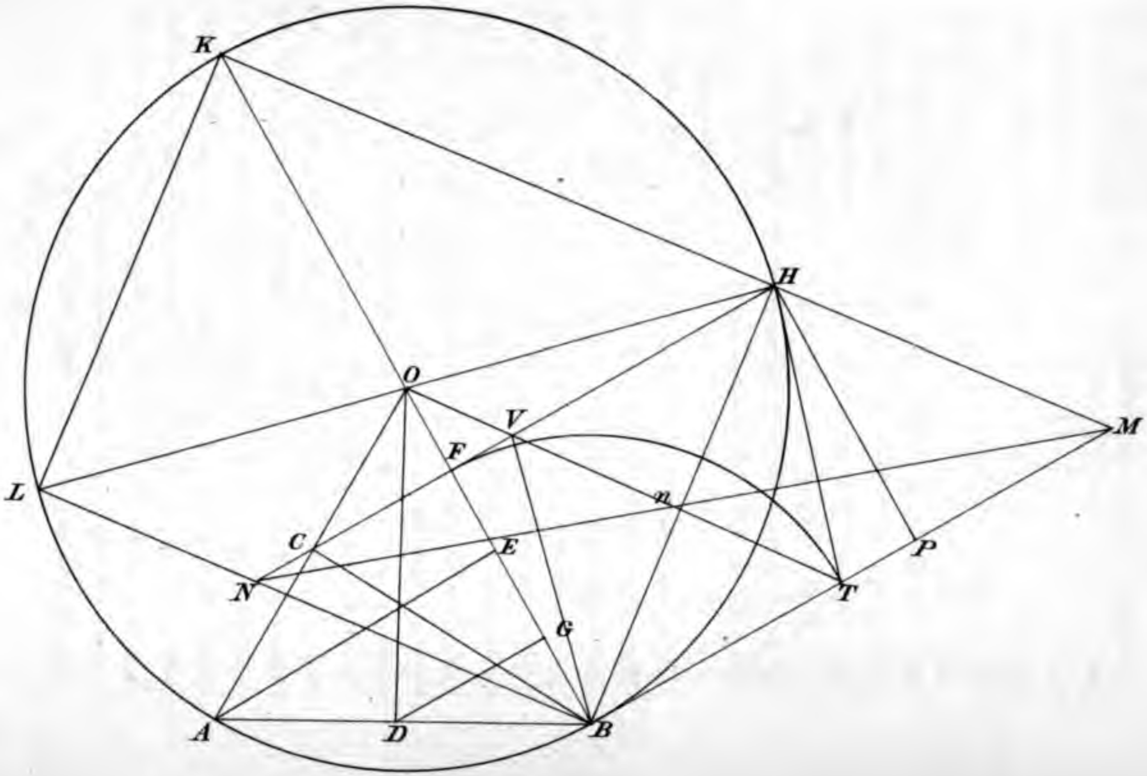
cession of diagrams, explanatory and demonstrative of the important discovery, that, *the eighth proposition of the sixth book of Euclid is inconsistent with the forty-seventh proposition of the first book; and that it is the former, not the latter, that is at fault.*

It subsequently occurred to me, that if Euclid could be *at fault* in one Theorem, he might be *at fault* in others, and upon further examination I discovered, that *the twelfth and thirteenth propositions of the second book of Euclid are also inconsistent with forty-seventh proposition of the first book, and again, that it is the former not the latter, that is at fault.*

In the correspondence with my friend, I directed his attention to a succession of geometrical figures, derived from the foregoing very simple diagram, but for my present purpose it is quite sufficient to take one of them.

In the annexed diagram, let A and B be two points dotted at random. Join A B. On A B describe the equilateral triangle O A B, and from the angles of the triangle draw straight lines, bisecting the angles and their subtending sides at the points C, D, and E. With O as centre, and O A or O B as interval, describe the circle. Bisect E O at F and E B at G. Join D G, and from the point F draw a straight line parallel to A E and D G, to meet and terminate in the circumference of the circle at the point H, and join F C. Produce B O to meet and terminate in the circumference of the circle at the point K, and join K H and H B, producing the right-angled triangle K H B. *(Euclid: Prop.* 31, *Book* 3.*)* From the angle B draw a straight line at right-angles to K B, and therefore tangental to the circle, to meet K H produced at M, constructing the right-angled triangle K B M. From the

DIAGRAM.



point H, let fall the perpendicular H P. From the angle H draw a straight line through the point O, the centre of the circle, to meet and terminate in the circumference at the point L, and join K L and L B, producing an inscribed right-angled parallelogram, K L B H, to the circle. Produce H C to meet and terminate in L B, a side of the parallelogram K L B H at the point N, and join N M. Bisect H N at V, and join V B. From the point O, the centre of the circle, draw a straight line through the point V, to meet and terminate in B M, the base of the right-angled triangle K B M, at the point T, and join T H. With B as centre and B F as interval, describe the arc F T.

Now, Sir, in this geometrical figure there are many things that will be — or, I should rather say, will appear to be — self-evident to any Mathematician. First: That B F = B T. Second: That the triangles K B M, K F H, and H P M are similar right-angled triangles. Third: Because K B is bisected at O, and B M at T; K B M and O B T are similar right-angled triangles. Fourth: Because K B and L H are diagonals of the parallelogram K L B H; K H B and B L K are similar right-angled triangles. Fifth: Because the diagonals of the parallelograms H N B M and H V B T intersect and bisect each other; H B, the diagonal common to both parallelograms, is bisected by O T, the hypothenuse of the right-angled triangle O B T. Sixth: H B is a diagonal of the right-angled parallelogram F B P H, and if we draw the other diagonal F P, these diagonals would also intersect and bisect each other at the point *n*, the point of intersection between the diagonals of the parallelograms H N B M and H V B T.

I shall not attempt to elaborate all the properties of this remarkable geometrical figure. I shall confine myself to demonstrating, by means of it, that *Euclid is at fault* in three of his most important theorems. This I shall do in the simplest way possible, and so bring my proofs within the capacity of anyone possessed of the most moderate geometrical and mathematical attainments.

Let K B, the diameter of the circle, = 8.

Then : By construction :

$$K\,H = \tfrac{4}{5}\,(K\,B) = 6{\cdot}4$$
$$H\,B = \tfrac{3}{4}\,(K\,H) = \tfrac{3}{5}\,(K\,B) = 4{\cdot}8$$
$$B\,M = \tfrac{3}{4}\,(K\,B) = \tfrac{3}{5}\,(K\,M) = 6.$$
$$H\,M = \tfrac{3}{4}\,(H\,B) = 3{\cdot}6.$$
$$K\,M = K\,H + H\,M = 10.$$
$$B\,F = B\,T = \tfrac{1}{2}\,(B\,M) = 3.$$
$$\text{And, } K\,F = K\,B - B\,F = 5.$$

Now, the triangles on each side of H B, are similar to the whole triangle K B M, and to each other ; and it follows of necessity, that $K\,B^2 - K\,H^2 = B\,M^2 - H\,M^2$; that is, the equation, $(8^2 - 6{\cdot}4^2) = (6 - 3{\cdot}6^2) = H\,B^2 = 23{\cdot}04$

But, K H B is a right-angled triangle, and H F is perpendicular to K B, by construction ; therefore, according to Euclid, Prop. 8, Book 6, $K\,H^2 - K\,F^2$ should equal $H\,B^2 - B\,F^2$, and both should equal $H\,F^2$. But this is not a fact.

For :

$$K\,H^2 - K\,F^2 = 6{\cdot}4^2 - 5^2 = 40{\cdot}96 - 25 = 15{\cdot}96.$$

But, $H\,B^2 - B\,F^2 = 4{\cdot}8^2 - 3^2 = 23{\cdot}04 - 9 = 14{\cdot}04.$

Therefore, it follows of necessity, that the triangles K F H and H F B are not similar to the whole triangle K H B, and to each other. Hence : the eighth proposition

of the sixth book of Euclid is not of general and universal application, and consequently, *is not true*, under all circumstances. (Q. E. D.)

Again: because H N is parallel to B M, B N parallel to H M, and H B a diagonal of the parallelogram H N B M, by construction; the triangles H B N and B H M are similiar and equal right-angled triangles, and H B is the perpendicular of, and common to, both. Now, when K B the diameter of the circle = 8, then H B = 4·8, B N = 3·6, and H N = 6. But, L B = K H, and K H = 6·4, when K B = 8; therefore, L B = 6·4. But, L B — N B = N L; therefore, 6·4 — 3·6 = 2·8 = N L.

Now, H B L is a right-angled triangle, and H N L, a part of it, is an oblique-angled triangle; and according to Euclid, Prop. 12, book 2, $\{HN^2 + NL^2 + 2(NL \cdot NB)\} = HL^2$; that is, $\{6^2 + 2{\cdot}8^2 + 2(2{\cdot}8 \times 3{\cdot}6)\} = (36 + 7{\cdot}84 + 20{\cdot}16) = 64 = HL^2$.

But H F K is a right-angled triangle, and H O K, a part of it, is an oblique-angled triangle, of which H O and O K sides of this triangle, are radii of the circle. But K F — O K = O F, by construction; and when K B the diameter of the circle = 8, H O and O K = 4, and O F = 1. Now, $\{HO^2 + OK^2 + 2(OK \cdot OF)\} = \{4^2 + 4^2 + 2(4 \times 1)\} = (16 + 16 + 8) = 40$. But, when K B = 8, K H = 6·4; therefore, $6{\cdot}4^2 = 40{\cdot}96 = KH^2$ and is greater than 40; that is greater than $\{HO^2 + OK^2 + 2(OK \cdot OF)\}$. Therefore, it follows of necessity, that the twelfth proposition of the second book of Euclid is not of general and universal application, and consequently *is not true*, under all circumstances. (Q. E. D.)

The 13th proposition of the second book of Euclid is simply the converse of the 12th; and it follows, that

if the 12th proposition be fallacious, the 15th must necessarily be so, and there is no occasion to burden my Letter with the calculations, to prove it.

In the right-angled triangles K F H and H F B, H F is the perpendicular, and common to both. Now to the Mathematician it will appear, that the triangles K F H, H F B, and K B M, are similiar right-angled triangles. But this is not so. For, if this were true, we should get the following analogy or proportion. K B : B M : : K F : F H that is, 8 : 6 : : 5 : 3·75. But, K H²—K F² is greater than 3·75, and therefore greater than F H : and H B²—B F² is less than 3·75, and therefore less than F H. Hence, K F H, H F B, and K B M are *not* similar triangles.

Where is the Mathematician who would have thought of doubting, that K F H and K B M are similar triangles? It is no doubt true that the angles in these triangles are similar, and one angle common to both : but this does not make them similar triangles.

For: If K F H and K B M were similar right-angled triangles, we should get the anology or proportion, K F : F B : : K H : H M. But, K F : F B : : K H : 3·84 ; that is, 5 : 3 : : 6·4 : 3·84 when K F + F B = 8. But, H M = 3·6 when the diameter of the circle = 8 ; therefore K F H and K B M *cannot* be similar right-angled triangles.

Again : We should get the analogy or proportion, K B : B M : : H P : P M ; that is, 8 : 6 : : 3 : 2·25. But, H P M is a right-angled triangle of which the sides H P and P M contain the right-angle ; therefore $HP^2 + PM^2 = 3^2 + 2{\cdot}25^2 = 9 + 5{\cdot}0625 = 14{\cdot}0625 = HM^2$; therefore, $\sqrt{14{\cdot}0625} = 3{\cdot}75 = HM$. But, I have proved that H M 3·6, when the diameter of the circle = 8 ; therefore,

K B M and H P M cannot be similar right-angled triangles.

The sincere and earnest geometrical enquirer will make many more remarkable discoveries, by means of this geometrical figure.

PROBLEM.

Construct a geometrical figure, in which there shall be two dissimilar and unequal right-angled triangles, of which the sides subtending the right-angle shall be equal.

This problem involves most important consequences to mathematical science, and I question if there be a living mathematician competent to solve it. This, you, Sir, can readily find out, through the President of the Mathematical and Physical Section of the British Association. Had that Association permitted me to bring my discoveries before the Physical Section, they would have had the solution of this problem, and the consequences involved in it before now.

It is my present determination never again to read a Paper, or attempt to read a Paper, at the *British Association.* At the last meeting, at Dundee, I offered, and was anxious, to read two Papers in the Mathematical and Physical Section ; but the Committee of that Section obstinately persisted in their determination not to permit me. The subject of these Papers was mean proportionals, and in one of them my object was to shew that from any given determinate quantity (it may be commensurable or incommensurable), we may obtain two pairs of numbers, of which the mean proportional of both pairs, shall be the same finite quantity. This is a discovery in Mathe-

matics, and leads to very important consequences. The *fact* is, the subject of mean proportionals may be reduced to a *theory,* and a most useful and valuable *theory* too, in Mathematical Science; and of this *fact* I believe our recognised *Mathematical Authorities* to be profoundly ignorant, at this moment. Is it not marvellous that, in an Association, called an Association for the Advancement of Science, the "guiding stars" if it should refuse to permit such subjects to be dealt with?

In conclusion: Strange as at first sight it may appear to you, Sir, for any one to assert that, *Euclid is at fault;* if you should have read so far, you will have discovered that, "*'tis not more strange than true.*" In bringing this fact under your notice, *I have done my duty;* and it remains for you, as the forthcoming President of "*The British Aassociation for the Advancement of Science,*" *and* as such, the guardian of its interests for a season, *to do yours.*

I am, Sir,

Yours very respectfully,

JAMES SMITH.

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